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EQUILIBRIUM POINTS OF A SINGULAR COOPERATIVE SYSTEM WITH FREE BOUNDARY

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Abstract. In this paper we initiate the study of maps minimising the energy

$$\int_D (|\nabla \mathbf{u}|^2 + 2|\mathbf{u}|)\, dx,$$

which, due to Lipschitz character of the integrand, gives rise to the singular Euler equations

$$\Delta \mathbf{u} = \frac{\mathbf{u}}{|\mathbf{u}|}\chi_{\{|\mathbf{u}|>0\}}, \qquad \mathbf{u} = (u_1, \cdots, u_m)\ .$$

Our primary goal in this paper is to set up a road map for future developments of the theory related to such energy minimising maps.

Our results here concern regularity of the solution as well as that of the free boundary. They are achieved by using monotonicity formulas and epiperimetric inequalities, in combination with geometric analysis.

1. Introduction

1.1. Background.

In this paper we shall study the singular system

$$\Delta \mathbf{u} = \tfrac{\mathbf{u}}{|\mathbf{u}|}\chi_{\{|\mathbf{u}|>0\}}, \qquad \mathbf{u} = (u_1, \cdots, u_m)\,, \tag{1}$$

where $\mathbf{u} : \mathbb{R}^n \supset D \to \mathbb{R}^m$, $n \geq 2$, $m \geq 1$, and $|\cdot|$ is the Euclidean norm on the respective spaces. System (1) is a particular example of the equilibrium state of a cooperative system: the corresponding reaction-diffusion system

$$u_t - \Delta u = -\frac{u}{\sqrt{u^2 + v^2}},$$
$$v_t - \Delta v = -\frac{v}{\sqrt{u^2 + v^2}}$$

would mean that, considering the concentrations u and v of two species/reactants, each species/reactant slows down the extinction/reaction of the other species. The special choice of our reaction kinetics would assure a *constant* decay/reaction rate in the case that u and v are of comparable size.

System (1) may also be seen as one of the simplest extensions of the classical *obstacle problem* to the vector-valued case: Solutions of the classical obstacle problem are minimisers of the energy $\int_D (\frac{1}{2}|\nabla u|^2 + \max(u, 0))\, dx$, where $u : \mathbb{R}^n \supset D \to \mathbb{R}$. Solutions of (1)

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are minimisers of the energy

$$\int_D (|\nabla \mathbf{u}|^2 + 2|\mathbf{u}|)\, dx. \tag{2}$$

It is noteworthy that in the scalar case, i.e. when $m = 1$, one recovers the two phase free boundary problem

$$\Delta u = \chi_{\{u>0\}} - \chi_{\{u<0\}},$$

contained in the analysis of [14]. While [14] as well as the two-phase result [6] relied essentially on the use of the monotonicity formula by Alt-Caffarelli-Friedman [4], a corresponding formula seems to be unavailable in our vector-valued problem.

There are several results concerning the obstacle problem for systems of various types: Optimal switching, multi-membranes, control of systems, constrained weakly elliptic systems, vector-valued obstacle problems, and probably many others. Although not directly relevant to our work, we refer to some papers that might be of interest for the readers [1], [2], [7], [8], [9], [10], [11].

1.2. **Main Result and Plan of the paper.** In this paper we are interested in qualitative behavior of the minimisers $\mathbf{u}$ of the functional (2) as well as of the free boundary $\partial\{x : |\mathbf{u}(x)| > 0\}$; here $\mathbf{u} = (u_1, \cdots, u_m)$ and $m \geq 1$. Note that the part of the free boundary where the gradient $\nabla \mathbf{u} \neq 0$ is, by the implicit function theorem, locally a $C^{1,\beta}$-surface, so that we are more concerned with the part where the gradient vanishes.

The main results of this paper (presented in Theorem 5) states that the set of "regular" free boundary points of the minimisers $\mathbf{u}$ to the functional (2) is locally a $C^{1,\beta}$ surface.

In proving this result we need an array of technical tools including monotonicity formulas (Lemma 1 in Section 4), quadratic growth of solutions (Theorem 2), and an epiperimetric inequality (Theorem 1), for the balanced energy functional (3).

An epiperimetric inequality has been proved in [19] by one of the authors for the scalar obstacle problem. See also [12] for a related approach to the scalar obstacle problem with Dini continuous coefficients.

1.3. **Notation.** Throughout this paper $\mathbb{R}^n, \mathbb{R}^m, \mathbb{R}^{nm}$ etc. will be equipped with the Euclidean inner product $x{\cdot}y$ and the induced norm $|x|$, $B_r(x^0)$ will denote the open n-dimensional ball of center x^0, radius r and volume $r^n\,\omega_n$, $B'_r(x^0) := \{x \in B_r(x^0) \,:\, x_n = (x^0)_n\}$, $B^+_r(x^0) := \{x \in B_r(x^0) \,:\, x_n > (x^0)_n\}$ and $\mathbf{e}^i$ the i-th unit vector in $\mathbb{R}^k$. If the center x^0 is not specified, then it is assumed to be the origin. Given a set $A \subset \mathbb{R}^n$, we denote its interior by A° and its characteristic function by χ_A. In the text we use the n-dimensional Lebesgue-measure $|A|$ of a set A and the k-dimensional Hausdorff-measure $\mathcal{H}^k$. When considering the boundary of a given set, ν will typically denote the topological outward normal to the boundary and $\nabla_\theta f := \nabla f \,-\, \nabla f \cdot \nu\,\nu$ the surface derivative of a given function f. Finally, we shall often use abbreviations for inverse images like $\{u > 0\} := \{x \in D \,:\, u(x) > 0\}$, $\{x_n > 0\} := \{x \in \mathbb{R}^n \,:\, x_n > 0\}$ etc. and occasionally we employ the decomposition $x = (x', x_n)$ of a vector $x \in \mathbb{R}^n$. Last, let $\Gamma(\mathbf{u}) := D \cap \partial\{x \in D : |\mathbf{u}(x)| > 0\}$ and $\Gamma_0(\mathbf{u}) := \Gamma(\mathbf{u}) \cap \{x : \nabla \mathbf{u}(x) = 0\}$.

2. The Epiperimetric Inequality

Following [19], we prove in this section an epiperimetric inequality, which tells us that close to half-plane solutions, the minimal energy achieved is lower than that of 2-homogeneous functions, and the energy difference can be estimated. This will imply in later sections a certain non-degeneracy of the energy close to half-plane solutions, and ultimately lead to regularity of the free boundary. Since the epiperimetric inequality is rather an abstract property of the energy, and represents the core of our result, we put this section at the beginning. Although the proof follows partly the proof in [19], the PDE resulting from the "linearization" carried out in the proof is different from that in [19] and introduces new difficulties.

Let

$$M(\mathbf{v}) := \int_{B_1} (|\nabla \mathbf{v}|^2 + 2|\mathbf{v}|) - 2\int_{\partial B_1} |\mathbf{v}|^2 \, d\mathcal{H}^{n-1}, \tag{3}$$

and let

$$\mathbb{H} := \{\frac{\max(x\cdot\nu,0)^2}{2}\mathbf{e} : \nu \text{ is a unit vector in } \mathbb{R}^n \text{ and } \mathbf{e} \text{ is a unit vector in } \mathbb{R}^m\}. \tag{4}$$

We define

$$\frac{\alpha_n}{2} := M(\frac{\max(x\cdot\nu,0)^2}{2}\mathbf{e}). \tag{5}$$

Theorem 1. *There exists* $\kappa \in (0,1)$ *and* $\delta > 0$ *such that if* $\mathbf{c}$ *is a homogeneous function of degree* 2 *satisfying* $\|\mathbf{c}-\mathbf{h}\|_{W^{1,2}(B_1;\mathbb{R}^m)} + \|\mathbf{c}-\mathbf{h}\|_{L^\infty(B_1;\mathbb{R}^m)} \le \delta$ *for some* $\mathbf{h}\in\mathbb{H}$*, then there is a* $\mathbf{v}\in W^{1,2}(B_1;\mathbb{R}^m)$ *such that* $\mathbf{v} = \mathbf{c}$ *on* ∂B_1 *and*

$$M(\mathbf{v}) \le (1-\kappa)M(\mathbf{c}) + \kappa\frac{\alpha_n}{2}. \tag{6}$$

Remark: Note that the closeness in L^∞ is not really necessary and is assumed only in order to avoid capacity arguments in the proof.

Proof of the Theorem. Suppose towards a contradiction that there are sequences $\kappa_k \to 0, \delta_k \to 0$, $\mathbf{c}_k \in W^{1,2}(B_1;\mathbb{R}^m)$, and $\mathbf{h}_k \in \mathbb{H}$ such that $\mathbf{c}_k$ is a homogeneous function of degree 2 and satisfies

$$\|\mathbf{c}_k-\mathbf{h}_k\|_{W^{1,2}(B_1;\mathbb{R}^m)} = \delta_k, \qquad \|\mathbf{c}_k-\mathbf{h}_k\|_{L^\infty(B_1;\mathbb{R}^m)} \to_{k\to\infty} 0$$

and that

$$M(\mathbf{v}) > (1-\kappa_k)M(\mathbf{c}_k) + \kappa_k\frac{\alpha_n}{2} \qquad \text{for all } \mathbf{v}\in \mathbf{c}_k + W^{1,2}_0(B_1;\mathbb{R}^m). \tag{7}$$

Rotating in $\mathbb{R}^n$ and in $\mathbb{R}^m$ if necessary we may assume that

$$\mathbf{h}_k(x) = \frac{\max(x_n,0)^2}{2}\mathbf{e}^1 =: \mathbf{h}$$

where $\mathbf{e}^1 = (1,0,\dots,0)\in\mathbb{R}^m$. Subtracting from (7) $M(\mathbf{h}) = \frac{\alpha_n}{2}$, we obtain

$$(1-\kappa_k)(M(\mathbf{c}_k)-M(\mathbf{h})) < M(\mathbf{v})-M(\mathbf{h}) \text{ for every } \mathbf{v}\in W^{1,2}(B_1;\mathbb{R}^m) \text{ such that } \mathbf{v}=\mathbf{c}_k \text{ on } \partial B_1. \tag{8}$$

Observe now that for each $\phi \in W^{1,2}(B_1)$ and $h := \frac{\max(x_n,0)^2}{2}$

$$2\int_{B_1}(\nabla h\cdot\nabla\phi + \chi_{\{x_n>0\}}\phi) - 2\int_{\partial B_1} 2h\phi\, d\mathcal{H}^{n-1} = 2\int_{\partial B_1}(\nabla h\cdot\nu - 2h)\phi\, d\mathcal{H}^{n-1} = 0,$$

and therefore

$$I := 2\int_{B_1}(\nabla\mathbf{h}\cdot\nabla(\mathbf{c}_k-\mathbf{h}) + \chi_{\{x_n>0\}}\mathbf{e}^1\cdot(\mathbf{c}_k-\mathbf{h})) - 2\int_{\partial B_1} 2\mathbf{h}\cdot(\mathbf{c}_k-\mathbf{h})\, d\mathcal{H}^{n-1} = 0.$$

Subtracting $(1-\kappa_k)I$ from the left-hand side of (8) and subtracting I with $\mathbf{c}_k$ replaced by $\mathbf{v}$ from the right-hand side of (8), we obtain thus

$$\begin{aligned}&(1-\kappa_k)\Big[\int_{B_1}(|\nabla\mathbf{c}_k|^2+2|\mathbf{c}_k|)-2\int_{\partial B_1}|\mathbf{c}_k|^2\,d\mathcal{H}^{n-1}-\int_{B_1}(|\nabla\mathbf{h}|^2+2|\mathbf{h}|)+2\int_{\partial B_1}|\mathbf{h}|^2\,d\mathcal{H}^{n-1}\\&-2\int_{B_1}(\nabla\mathbf{h}\cdot\nabla(\mathbf{c}_k-\mathbf{h})+\chi_{\{x_n>0\}}\mathbf{e}^1\cdot(\mathbf{c}_k-\mathbf{h}))+2\int_{\partial B_1}2\mathbf{h}\cdot(\mathbf{c}_k-\mathbf{h})\,d\mathcal{H}^{n-1}\Big]\\&<\int_{B_1}(|\nabla\mathbf{v}|^2+2|\mathbf{v}|)-2\int_{\partial B_1}|\mathbf{v}|^2\,d\mathcal{H}^{n-1}-\int_{B_1}(|\nabla\mathbf{h}|^2+2|\mathbf{h}|)+2\int_{\partial B_1}|\mathbf{h}|^2\,d\mathcal{H}^{n-1}\\&-2\int_{B_1}(\nabla\mathbf{h}\cdot\nabla(\mathbf{v}-\mathbf{h})+\chi_{\{x_n>0\}}\mathbf{e}^1\cdot(\mathbf{v}-\mathbf{h}))+2\int_{\partial B_1}2\mathbf{h}\cdot(\mathbf{v}-\mathbf{h})\,d\mathcal{H}^{n-1}.\end{aligned}$$

Rearranging terms yields

$$\begin{aligned}(9)\quad&(1-\kappa_k)\Big[\int_{B_1}|\nabla(\mathbf{c}_k-\mathbf{h})|^2-2\int_{\partial B_1}|\mathbf{c}_k-\mathbf{h}|^2\,d\mathcal{H}^{n-1}+2\int_{B_1^-}|\mathbf{c}_k|+2\int_{B_1^+}(|\mathbf{c}_k|-\mathbf{e}^1\cdot\mathbf{c}_k)\Big]\\&<\int_{B_1}|\nabla(\mathbf{v}-\mathbf{h})|^2-2\int_{\partial B_1}|\mathbf{v}-\mathbf{h}|^2\,d\mathcal{H}^{n-1}+2\int_{B_1^-}|\mathbf{v}|+2\int_{B_1^+}(|\mathbf{v}|-\mathbf{e}^1\cdot\mathbf{v}).\end{aligned}$$

Define now the sequence of functions $\mathbf{w}_k := (\mathbf{c}_k-\mathbf{h})/\delta_k$. Then $\|\mathbf{w}_k\|_{W^{1,2}(B_1;\mathbb{R}^m)} = 1$ and, passing to a subsequence if necessary, $\mathbf{w}_k \to \mathbf{w}$ weakly in $W^{1,2}(B_1;\mathbb{R}^m)$. In order to obtain a contradiction, we are going to prove that $\mathbf{w}_k \to \mathbf{w}$ strongly in $W^{1,2}(B_1;\mathbb{R}^m)$ and that $\mathbf{w}\equiv 0$ in $B_1(0)$.

Step 1: $\mathbf{w}\equiv 0$ in B_1^-, and $\int_{B_1^+}(|\mathbf{c}_k|-\mathbf{e}^1\cdot\mathbf{c}_k)\le C\delta_k^2$

Plug in $\mathbf{v} := (1-\zeta)\mathbf{c}_k+\zeta\mathbf{h}$ in (9), where $\zeta\in W_0^{1,2}(B_1)$ is radial symmetric and satisfies $0\le\zeta\le 1$. Since $(\mathbf{v}-\mathbf{h})/\delta_k = (1-\zeta)\mathbf{w}_k$, we obtain

$$\begin{aligned}&2(1-\kappa_k)\left[\int_{B_1^-}\frac{|\mathbf{c}_k|}{\delta_k^2}+\int_{B_1^+}\frac{|\mathbf{c}_k|-\mathbf{e}^1\cdot\mathbf{c}_k}{\delta_k^2}\right]\\&<C_1+2\int_{B_1^-}(1-\zeta)\frac{|\mathbf{c}_k|}{\delta_k^2}+2\int_{B_1^+}(1-\zeta)\frac{|\mathbf{c}_k|-\mathbf{e}^1\cdot\mathbf{c}_k}{\delta_k^2}\end{aligned}$$

and

$$\int_{B_1^-}(\zeta-\kappa_k)\frac{|\mathbf{c}_k|}{\delta_k^2}+\int_{B_1^+}(\zeta-\kappa_k)\frac{|\mathbf{c}_k|-\mathbf{e}^1\cdot\mathbf{c}_k}{\delta_k^2}\le C_1.$$

Using the homogeneity of $\mathbf{c}_k$ we see that for large k,

$$\int_{B_1^-}(\zeta-\kappa_k)|\mathbf{c}_k| = \int_0^1(\zeta(\rho)-\kappa_k)\rho^{n+1}\,d\rho\int_{\{x_n<0\}\cap\partial B_1}|\mathbf{c}_k|\,d\mathcal{H}^{n-1}\ge c_0\int_{\{x_n<0\}\cap\partial B_1}|\mathbf{c}_k|\,d\mathcal{H}^{n-1},$$

where $c_0>0$ depends only on ζ and n. We also get the corresponding estimate in B_1^+. It follows that

$$(10)\qquad \int_{B_1^-}|\mathbf{c}_k|\le C_2\delta_k^2 \text{ and that } \int_{B_1^+}\Big(|\mathbf{c}_k|-\mathbf{e}^1\cdot\mathbf{c}_k\Big)\le C_2\delta_k^2.$$

In particular,

$$\int_{B_1^-}|\mathbf{w}_k|\le C_2\delta_k,$$

implying the statement of Step 1.

Step 2: $\Delta(\mathbf{e}^1\cdot\mathbf{w}) = 0$ in $B_1^+(0)$, $\mathbf{e}^j\cdot\mathbf{w} = d_j h$ in $B_1^+(0)$ for each $j>1$, and some constant d_j.

Fix a ball $B \subset\subset B_1^+$ and plug $\mathbf{v} := (1-\zeta)\mathbf{c}_k + \zeta(\mathbf{h} + \delta_k \mathbf{g})$ into (9), where $\zeta \in C_0^\infty(B_1^+)$ and $\mathbf{g} \in W^{1,2}(B_1;\mathbb{R}^m)$ such that $\zeta \equiv 1$ in B, $\zeta \equiv 0$ in B_1^- and $\mathbf{g}$ is a bounded $W^{1,2}(B_1;\mathbb{R}^m)$-function. Observing that

$$\frac{\mathbf{v}-\mathbf{h}}{\delta_k} = (1-\zeta)\frac{\mathbf{c}_k - \mathbf{h}}{\delta_k} + \zeta \mathbf{g},$$

we obtain —using (10) as well as the fact that $\mathbf{supp}\,\zeta \subset\subset B_1^+$— that

$$\begin{aligned}
&\int_{B_1^+} (2\zeta - \zeta^2)|\nabla \mathbf{w}_k|^2 + \frac{2}{\delta_k^2}\int_{B_1^+} \zeta(|\mathbf{c}_k| - \mathbf{e}^1 \cdot \mathbf{c}_k)\\
&\le o(1) + \int_{B_1^+} \zeta^2 |\nabla \mathbf{g}|^2 + 2\int_{B_1^+\setminus B} \Big(|\nabla\zeta|^2|\mathbf{g}-\mathbf{w}_k|^2 + (2\zeta - 2\zeta^2)\nabla \mathbf{w}_k \cdot \nabla \mathbf{g}\Big)\\
&+ 2\int_{B_1^+\setminus B} ((1-\zeta)\nabla\zeta \cdot \nabla \mathbf{w}_k(\mathbf{g}-\mathbf{w}_k) + \zeta\nabla\zeta\cdot\nabla\mathbf{g}(\mathbf{g}-\mathbf{w}_k))\\
&+ \frac{2}{\delta_k^2}\int_{B_1^+} \zeta\Big(|\mathbf{h}+\delta_k\mathbf{g}| - \mathbf{e}^1\cdot(\mathbf{h}+\delta_k\mathbf{g})\Big).
\end{aligned}$$

Note that $\delta_k \mathbf{w}_k \to 0$ uniformly in B_1. Therefore we have on $\mathbf{supp}\,\zeta$

$$|\mathbf{c}_k| - \mathbf{e}^1\cdot\mathbf{c}_k = (h + \delta_k \mathbf{e}^1\cdot\mathbf{w}_k)\left(\sqrt{1+\delta_k^2\frac{|\mathbf{w}_k|^2 - (\mathbf{e}^1\cdot\mathbf{w}_k)^2}{(h+\delta_k\mathbf{e}^1\cdot\mathbf{w}_k)^2}} - 1\right) = o(\delta_k^2) + \frac{\delta_k^2}{2}\frac{|\mathbf{w}_k|^2 - (\mathbf{e}^1\cdot\mathbf{w}_k)^2}{h+\delta_k\mathbf{e}^1\cdot\mathbf{w}_k}$$

and similarly

$$|\mathbf{h}+\delta_k\mathbf{g}| - \mathbf{e}^1\cdot(\mathbf{h}+\delta_k\mathbf{g}) = o(\delta_k^2) + \frac{\delta_k^2}{2}\frac{|\mathbf{g}|^2 - \left(\mathbf{e}^1\cdot\mathbf{g}\right)^2}{h+\delta_k\mathbf{e}^1\cdot\mathbf{g}}.$$

Letting $k\to\infty$ we may then drop the assumption that $\mathbf{g}$ is bounded. In particular, for $\mathbf{g}$ such that $\mathbf{g} = \mathbf{w}$ in $B_1 \setminus B$, we arrive at the inequality

$$\int_B |\nabla\mathbf{w}|^2 + \int_B \frac{|\mathbf{w}|^2 - (\mathbf{e}^1\cdot\mathbf{w})^2}{h} \le \int_B |\nabla\mathbf{g}|^2 + \int_B \frac{|\mathbf{g}|^2 - (\mathbf{e}^1\cdot\mathbf{g})^2}{h}$$

for all $\mathbf{g} \in W^{1,2}(B_1;\mathbb{R}^m)$ coinciding with $\mathbf{w}$ on ∂B.

Calculation of the first variation yields that

$$\begin{aligned}
&\Delta(\mathbf{e}^1\cdot\mathbf{w}) = 0 \quad \text{in} \quad B,\\
&\Delta(\mathbf{e}^j\cdot\mathbf{w}) = \frac{\mathbf{e}^j\cdot\mathbf{w}}{h} \quad \text{in} \quad B \quad \text{for} \quad j>1.
\end{aligned}$$

By Lemma 4 as well as the homogeneity of $\mathbf{w}$ and the fact that $\mathbf{w}\equiv 0$ in B_1^- we obtain

$$\mathbf{e}^j\cdot\mathbf{w}(x) = d_j h(x) \text{ for each } j>1,$$

where d_j is a constant real number.

Step 3: $w := \mathbf{e}^1\cdot\mathbf{w} = 0$ **in** B_1**.**

As w is harmonic in B_1^+, homogeneous of degree 2 and satisfies $w = 0$ in B_1^- we obtain (using for example odd reflection and the Liouville theorem) that $w(x) = \sum_{j=1}^{n-1} a_{nj}x_j x_n$ in B_1^+. Remember that we have chosen $\mathbf{h}$ as the minimiser of $\inf_{\mathbf{h}\in\mathbb{H}} \|\mathbf{c}_k - \mathbf{h}\|_{W^{1,2}(B_1;\mathbb{R}^m)}$. It follows that for $\mathbf{h}_\nu := \mathbf{e}^1\max(x\cdot\nu,0)^2/2$,

$$\frac{(\mathbf{w}_k,\mathbf{h}_\nu - \mathbf{h})_{W^{1,2}(B_1;\mathbb{R}^m)}}{|\nu - \mathbf{e}^n|} \le \frac{1}{2\delta_k}\frac{\|\mathbf{h}_\nu - \mathbf{h}\|^2_{W^{1,2}(B_1;\mathbb{R}^m)}}{|\nu-\mathbf{e}^n|} \to 0 \text{ as } \nu\to\mathbf{e}^n.$$

Therefore

$$o(1) \geq \int_{B_1} \mathbf{w}_k \cdot \mathbf{e}^1 \Bigg[\chi_{\{x_n>0\}\cap\{x\cdot\nu>0\}} \frac{(x\cdot\nu)^2 - (x\cdot\mathbf{e}^n)^2}{|\nu-\mathbf{e}^n|} - \chi_{\{x_n>0\}\cap\{x\cdot\nu\leq 0\}} \frac{(x\cdot\mathbf{e}^n)^2}{|\nu-\mathbf{e}^n|}$$

$$+ \chi_{\{x_n\leq 0\}\cap\{x\cdot\nu>0\}} \frac{(x\cdot\nu)^2}{|\nu-\mathbf{e}^n|} \Bigg]$$

$$+ \int_{B_1} \Bigg[\chi_{\{x_n>0\}\cap\{x\cdot\nu>0\}} \frac{x\cdot(\nu+\mathbf{e}^n)(\nu-\mathbf{e}^n) + x\cdot(\nu-\mathbf{e}^n)(\nu+\mathbf{e}^n)}{|\nu-\mathbf{e}^n|}$$

$$- \chi_{\{x_n>0\}\cap\{x\cdot\nu\leq 0\}} \frac{2x_n\mathbf{e}^n}{|\nu-\mathbf{e}^n|} + \chi_{\{x_n\leq 0\}\cap\{x\cdot\nu>0\}} \frac{2x\cdot\nu\nu}{|\nu-\mathbf{e}^n|} \Bigg] \cdot \nabla\mathbf{w}_k \cdot \mathbf{e}^1.$$

Setting $\xi := \lim_{\nu\to\mathbf{e}^n} \frac{\nu-\mathbf{e}^n}{|\nu-\mathbf{e}^n|}$, we see that for $\nu \to \mathbf{e}^n$

$$\frac{(x\cdot\nu)^2 - (x\cdot\mathbf{e}^n)^2}{|\nu-\mathbf{e}^n|} \to 2x_n x\cdot\xi, \ \frac{x\cdot(\nu+\mathbf{e}^n)(\nu-\mathbf{e}^n)}{|\nu-\mathbf{e}^n|} \to 2x_n\xi, \ \frac{x\cdot(\nu-\mathbf{e}^n)(\nu+\mathbf{e}^n)}{|\nu-\mathbf{e}^n|} \to 2x\cdot\xi\mathbf{e}^n.$$

On the other hand, on the set $(\{x_n > 0\}\cap\{x\cdot\nu \leq 0\})\cup(\{x_n \leq 0\}\cap\{x\cdot\nu > 0\})$, $|x\cdot\nu| = O(|\nu-\mathbf{e}^n|)$ and $|x\cdot\mathbf{e}^n| = O(|\nu-\mathbf{e}^n|)$ as $\nu \to \mathbf{e}^n$. Passing first to the limit $\nu \to \mathbf{e}^n$ we conclude that

$$o(1) \geq 2\int_{B_1} [\mathbf{w}_k \cdot \mathbf{e}^1 x\cdot\xi \max(x_n, 0) + (\max(x_n, 0)\xi + \chi_{\{x_n>0\}} x\cdot\xi\mathbf{e}^n) \cdot \nabla\mathbf{w}_k \cdot \mathbf{e}^1].$$

Passing next to the limit $k \to \infty$, and taking into account that $\xi_n = 0$ and that

$$\nabla w = \begin{pmatrix} a_{nj}x_n \\ \sum_{j=1}^{n-1} a_{nj}x_j \end{pmatrix},$$

we obtain that

$$0 \geq \sum_{j=1}^{n-1} a_{nj} \int_{B_1} [\max(x_n, 0)^2 x\cdot\xi x_j + \max(x_n, 0)^2 \xi_j + \chi_{\{x_n>0\}} x\cdot\xi x_j]. \tag{11}$$

Since also

$$\int_{B_1} x_j x_i = 0 \qquad \text{for } i \neq j,$$

we deduce from (11) that

$$0 \geq \sum_{j=1}^{n-1} a_{nj}\xi_j \int_{B_1^+} (x_n^2 x_j^2 + x_n^2 + x_j^2) \qquad \text{for every } \xi = (\xi_1, \ldots, \xi_{n-1}, 0). \tag{12}$$

Thus $a_{nj} = 0$ for $j = 1, \ldots, n-1$, that is $w \equiv 0$ in B_1^+.

Step 4: $d_j = 0$ **for each** $j \geq 2$.

From Step 2-3 we know that $\mathbf{w}_k = \mathbf{d}h + \mathbf{z}_k$, where $\mathbf{d}\cdot\mathbf{e}^1 = 0$ and $\mathbf{z}_k \to 0$ weakly in $W^{1,2}(B_1; \mathbb{R}^m)$ as $k \to \infty$. It follows that $\mathbf{c}_k = h(\mathbf{e}^1 + \delta_k\mathbf{d}) + \delta_k\mathbf{z}_k$. By assumption,

$$1 = \|\mathbf{d}h + \mathbf{z}_k\|^2_{W^{1,2}(B_1;\mathbb{R}^m)} = |\mathbf{d}|^2\|h\|^2_{W^{1,2}(B_1;\mathbb{R}^m)} + 2(\mathbf{d}h, \mathbf{z}_k)_{W^{1,2}(B_1;\mathbb{R}^m)} + \|\mathbf{z}_k\|^2_{W^{1,2}(B_1;\mathbb{R}^m)}. \tag{13}$$

Remember that we have chosen $\mathbf{h}$ as the minimiser of $\inf_{\mathbf{f}\in\mathbb{H}} \|\mathbf{c}_k - \mathbf{f}\|_{W^{1,2}(B_1;\mathbb{R}^m)}$. It follows that for $\mathbf{f} := h(\mathbf{e}^1 + \delta_k \mathbf{d})/\sqrt{1+\delta_k^2|\mathbf{d}|^2} \in \mathbb{H}$,

$$\delta_k = \|\mathbf{c}_k - h\mathbf{e}^1\|_{W^{1,2}(B_1;\mathbb{R}^m)} \le \|\mathbf{c}_k - \mathbf{f}\|_{W^{1,2}(B_1;\mathbb{R}^m)} = \left\| h(\mathbf{e}^1 + \delta_k \mathbf{d}) + \delta_k \mathbf{z}_k - \frac{h(\mathbf{e}^1 + \delta_k \mathbf{d})}{\sqrt{1+\delta_k^2|\mathbf{d}|^2}} \right\|_{W^{1,2}(B_1;\mathbb{R}^m)}$$

$$= \left\| \delta_k \mathbf{z}_k + \frac{h(\mathbf{e}^1 + \delta_k \mathbf{d})}{\sqrt{1+\delta_k^2|\mathbf{d}|^2}} \left(\sqrt{1+\delta_k^2|\mathbf{d}|^2} - 1\right) \right\|_{W^{1,2}(B_1;\mathbb{R}^m)} \le \delta_k \|\mathbf{z}_k\|_{W^{1,2}(B_1;\mathbb{R}^m)} + C_3 \delta_k^2 |\mathbf{d}|^2.$$

$$\text{Hence, } 1 \le \|\mathbf{z}_k\|_{W^{1,2}(B_1;\mathbb{R}^m)} + C_3 \delta_k |\mathbf{d}|^2. \tag{14}$$

Combining (13) and (14), we obtain that

$$|\mathbf{d}|^2 \|h\|^2_{W^{1,2}(B_1;\mathbb{R}^m)} + 2(\mathbf{d}h, \mathbf{z}_k)_{W^{1,2}(B_1;\mathbb{R}^m)} + \|\mathbf{z}_k\|^2_{W^{1,2}(B_1;\mathbb{R}^m)} \le \|\mathbf{z}_k\|^2_{W^{1,2}(B_1;\mathbb{R}^m)} + O(\delta_k).$$

Letting $k \to \infty$, we conclude that $|\mathbf{d}|^2 \|h\|^2_{W^{1,2}(B_1)} = 0$ and that $|\mathbf{d}| = 0$.

Step 5: $\mathbf{w}_k \to \mathbf{w}$ strongly in $W^{1,2}(B_1;\mathbb{R}^m)$.

Plug in $\mathbf{v} := (1-\zeta)\mathbf{c}_k + \zeta \mathbf{h}$ in (9), where $\zeta(x) = \min(2\max(1-|x|, 0), 1)$. Then

$$\frac{\mathbf{v} - \mathbf{h}}{\delta_k} = (1-\zeta)\mathbf{w}_k,$$

and we obtain that

$$(1-\kappa_k)\Big[\int_{B_1} |\nabla \mathbf{w}_k|^2 - 2\int_{\partial B_1} |\mathbf{w}_k|^2 \, d\mathcal{H}^{n-1} + 2\int_{B_1^-} \frac{|\mathbf{c}_k|}{\delta_k^2} + 2\int_{B_1^+} \frac{|\mathbf{c}_k| - \mathbf{e}^1 \cdot \mathbf{c}_k}{\delta_k^2} \Big]$$
$$< \int_{B_1} |\nabla((1-\zeta)\mathbf{w}_k)|^2 - 2\int_{\partial B_1} |(1-\zeta)\mathbf{w}_k|^2 \, d\mathcal{H}^{n-1} + 2\int_{B_1^-} \frac{(1-\zeta)|\mathbf{c}_k|}{\delta_k^2}$$
$$+ 2\int_{B_1^+} \frac{|(1-\zeta)\mathbf{c}_k + \zeta \mathbf{h}|}{\delta_k^2} - 2\int_{B_1^+} \frac{(1-\zeta)\mathbf{c}_k \cdot \mathbf{e}^1 + \zeta h}{\delta_k^2}.$$

Using the definition of ζ, it follows that

$$\int_{B_{1/2}} |\nabla \mathbf{w}_k|^2 \le C_4 \kappa_k + \int_{B_1} \left(|\nabla \zeta|^2 |\mathbf{w}_k|^2 - 2(1-\zeta)\nabla\zeta \cdot \nabla \mathbf{w}_k \mathbf{w}_k \right).$$

The integral on the left-hand side equals by homogeneity of $\mathbf{w}_k$

$$2^{-n-2} \int_{B_1} |\nabla \mathbf{w}_k|^2,$$

so that

$$\int_{B_1} |\nabla \mathbf{w}_k|^2 \le 2^{n+2} \left(C_4 \kappa_k + \int_{B_1} \left(|\nabla\zeta|^2 |\mathbf{w}_k|^2 - 2(1-\zeta)\nabla\zeta \cdot \nabla \mathbf{w}_k \cdot \mathbf{w}_k \right) \right) \to 0 \text{ as } k \to \infty.$$

Altogether we obtain a contradiction from $\mathbf{w} \equiv 0$, the strong convergence of $\mathbf{w}_k$ as well as the fact that $\|\mathbf{w}_k\|_{W^{1,2}(B_1;\mathbb{R}^m)} = 1$. □

3. Introduction to the problem and technical tools

Let D be a bounded Lipschitz domain in $\mathbb{R}^n$ and let $\mathbf{u} = (u_1, \ldots, u_m)$ be a minimiser of

$$E(\mathbf{u}) := \int_D (|\nabla \mathbf{u}|^2 + 2|\mathbf{u}|)$$

in the affine subspace $\{\mathbf{v} \in W^{1,2}(D; \mathbb{R}^m) : \mathbf{v} = \mathbf{u}_D \text{ on } \partial D\}$. Note that non-negativity, convexity and lower semicontinuity with respect to weak convergence imply existence of a minimiser for each $\mathbf{u}_D \in W^{1,2}(D; \mathbb{R}^m)$.

In order to compute the first variation of the energy, we compute for $\boldsymbol{\phi} \in W^{1,2}_0(D; \mathbb{R}^m)$

$$\begin{aligned} 0 &\le \epsilon \int_D 2\nabla \mathbf{u} \cdot \nabla \boldsymbol{\phi} + \epsilon^2 \int_D |\nabla \boldsymbol{\phi}|^2 + 2 \int_D (|\mathbf{u} + \epsilon \boldsymbol{\phi}| - |\mathbf{u}|) \\ &\le \epsilon \int_D 2\nabla \mathbf{u} \cdot \nabla \boldsymbol{\phi} + \epsilon^2 \int_D |\nabla \boldsymbol{\phi}|^2 + 2|\epsilon| \int_D |\boldsymbol{\phi}|. \end{aligned} \tag{15}$$

Dividing by ϵ and letting $\epsilon \to 0$, it follows that

$$\left| \int_D \nabla \mathbf{u} \cdot \nabla \boldsymbol{\phi} \right| \le \|\boldsymbol{\phi}\|_{L^1(D;\mathbb{R}^m)},$$

so that $\Delta \mathbf{u} \in L^\infty(D; \mathbb{R}^m)$. Applying standard L^p- and C^α-theory, we obtain that $\mathbf{u} \in W^{2,p}_{\text{loc}}(D; \mathbb{R}^m) \cap C^{1,\alpha}_{\text{loc}}(D; \mathbb{R}^m)$ for each $p \in [1, +\infty)$ and each $\alpha \in (0, 1)$. We see that $\Delta \mathbf{u} = 0$ a.e. in $\{\mathbf{u} = 0\}$. Moreover, in the open set $\{|\mathbf{u}| > \delta > 0\}$, passing to the limit in (15) yields

$$\Delta \mathbf{u} = \frac{\mathbf{u}}{|\mathbf{u}|} \text{ in } \{|\mathbf{u}| > \delta > 0\}.$$

Altogether we obtain that $\mathbf{u}$ is a strong solution of the equation

$$\Delta \mathbf{u} = \frac{\mathbf{u}}{|\mathbf{u}|} \chi_{\{|\mathbf{u}|>0\}}$$

in D.

Note that any other solution $\mathbf{v} \in W^{1,2}(D; \mathbb{R}^m)$ with the same boundary data $\mathbf{u}_D$ and satisfying the weak equation

$$\int_D \left(\nabla \mathbf{v} \cdot \nabla \boldsymbol{\phi} + \boldsymbol{\phi} \frac{\mathbf{v}}{|\mathbf{v}|} \chi_{\{|\mathbf{v}|>0\}} \right) = 0 \text{ for every } \boldsymbol{\phi} \in W^{1,2}_0(D; \mathbb{R}^m)$$

must coincide with $\mathbf{u}$: subtracting the equation for $\mathbf{u}$ and plugging in $\boldsymbol{\phi} := \mathbf{v} - \mathbf{u}$ yields

$$\int_D |\nabla(\mathbf{u} - \mathbf{v})|^2 \le - \int_D \left(\frac{\mathbf{u}}{|\mathbf{u}|} \chi_{\{|\mathbf{u}|>0\}} - \frac{\mathbf{v}}{|\mathbf{v}|} \chi_{\{|\mathbf{v}|>0\}} \right) \cdot (\mathbf{u} - \mathbf{v}) \le 0.$$

Thus the weak solution is unique and equals the minimiser of the problem, so that it is sufficient to consider minimisers.

Note that in contrast to the classical (scalar) obstacle problem, it is an open problem whether $\mathbf{u} \in W^{2,\infty}_{\text{loc}}(D; \mathbb{R}^m)$.

Remark 1. *Using standard elliptic theory combined with the estimate* $|\Delta \mathbf{u}| \le 1$ *we obtain that*

$$\sup_{B_{3/4}} |\mathbf{u}| + \sup_{B_{3/4}} |\nabla \mathbf{u}| \le C_1(n, m) \left(\|\mathbf{u}\|_{L^1(B_1;\mathbb{R}^m)} + 1 \right). \tag{16}$$

Remark 2. *If a sequence of solutions of our system* $\mathbf{u}_k$ *converges weakly in* $W^{1,2}(D; \mathbb{R}^m)$ *to* $\mathbf{u}$*, then Rellich's theorem together with the fact that* $D^2\mathbf{u} = 0$ *a.e. in* $\{\mathbf{u} = 0\}$*, implies that* $\mathbf{u}$ *is a solution, too.*

Proposition 1 (Non-Degeneracy). *Let* $\mathbf{u}$ *be a solution of (1) in* D. *If* $x^0 \in \overline{\{|\mathbf{u}| > 0\}}$ *and* $B_r(x^0) \subset D$, *then*

$$\sup_{B_r(x^0)} |\mathbf{u}| \geq \frac{1}{2n} r^2.$$

Proof. It is sufficient to prove a uniform estimate for $x^0 \in \{|\mathbf{u}| > 0\}$. Let $U(x) := |\mathbf{u}(x)|$. Then

$$\Delta U = 1 + \frac{A}{U} \text{ in } \{|\mathbf{u}| > 0\}, \text{ where } A = |\nabla \mathbf{u}|^2 - |\nabla U|^2 \geq 0. \tag{17}$$

Assuming $\sup_{B_r(x^0)} |\mathbf{u}| \leq \frac{1}{2n} r^2$, we obtain that the function

$$v(x) := U(x) - U(x^0) - \frac{1}{2n}|x - x^0|^2$$

is subharmonic in the connected component of $B_r(x^0) \cap \{|\mathbf{u}| > 0\}$ containing x^0, that $v < 0$ on the boundary of that component and that $v(x^0) = 0$, contradiction. □

Proposition 2. *Let* $\mathbf{u}$ *be a solution of (1) in* $B_1(0)$ *such that* $\|\mathbf{u} - \mathbf{h}\|_{L^1(B_1;\mathbb{R}^m)} \leq \epsilon < 1$, *where* $\mathbf{h} := \frac{\max(x_n,0)^2}{2} \mathbf{e}^1$. *Then*

$$B_{1/2}(0) \cap \mathbf{supp}\, \mathbf{u} \subset \left\{ x_n > -C\epsilon^{\frac{1}{2n+2}} \right\}$$

with a constant $C = C(n, m)$.

Proof. Suppose that $B_{1/2} \cap \{|\mathbf{u}| > 0\} \ni x^0$ and that $x^0_n = -\rho < 0$. It follows that

$$\|\mathbf{u}\|_{L^1(B_\rho(x^0);\mathbb{R}^m)} \leq \|\mathbf{u} - \mathbf{h}\|_{L^1(B_1;\mathbb{R}^m)} \leq \epsilon.$$

By the non-degeneracy property Proposition 1 we know that

$$|\mathbf{u}(y)| = \sup_{B_{\rho/2}(x^0)} |\mathbf{u}| \geq \frac{1}{8n}\rho^2$$

for some $y \in B_{\frac{\rho}{2}}(x^0)$. From Remark 1 we infer that

$$\inf_{B_{\sigma\rho^2}(y)} |\mathbf{u}| \geq \frac{1}{8n}\rho^2 - 2C_1(n,m)\sigma\rho^2 \geq \frac{1}{16\,n}\rho^2,$$

provided that σ has been chosen small enough, depending only on n and m. Combining our estimates, we obtain that

$$\epsilon \geq \|\mathbf{u}\|_{L^1(B_{\sigma\rho^2}(y);\mathbb{R}^m)} \geq \left(\frac{1}{16\,n}\rho^2\right)|B_1|\left(\sigma\rho^2\right)^n,$$

a contradiction, if $\epsilon < C_2(n,m)\rho^{2n+2}$. It follows that $|\mathbf{u}(x)| = 0$ for $x_n \leq -C(n,m)\epsilon^{\frac{1}{2n+2}}$. □

4. Monotonicity Formula and Consequences

Lemma 1. *Let* $\mathbf{u}$ *be a solution of (1) in* $B_{r_0}(x^0)$ *and let*

$$W(\mathbf{u}, x^0, r) = \frac{1}{r^{n+2}} \int_{B_r(x^0)} (|\nabla \mathbf{u}|^2 + 2|\mathbf{u}|) - \frac{2}{r^{n+3}} \int_{\partial B_r(x^0)} |\mathbf{u}|^2 \, d\mathcal{H}^{n-1}.$$

For $0 < r < r_0$,

$$\frac{dW(\mathbf{u}, x^0, r)}{dr} = 2 \int_{\partial B_1(0)} r \left| \frac{d}{dr} \mathbf{u}_r \right|^2 \, d\mathcal{H}^{n-1},$$

where $\mathbf{u}_r(x) = \mathbf{u}(rx + x^0)/r^2$.

Proof. The proof of this lemma follows by now standard arguments of G.S. Weiss (see [18] and [19]). A short proof consists in scaling

$$\begin{aligned}
\frac{d}{dr}W(\mathbf{u},x^0,r) &= \frac{d}{dr}\left(\int_{B_1(0)}(|\nabla\mathbf{u}_r|^2+2|\mathbf{u}_r|) - 2\int_{\partial B_1(0)}|\mathbf{u}_r|^2\,d\mathcal{H}^{n-1}\right)\\
&= 2\int_{B_1(0)}(\nabla\mathbf{u}_r\cdot\frac{d}{dr}\nabla\mathbf{u}_r+\frac{\mathbf{u}_r}{|\mathbf{u}_r|}\cdot\frac{d}{dr}\mathbf{u}_r) - 2\int_{\partial B_1(0)}2\mathbf{u}_r\cdot\frac{d}{dr}\mathbf{u}_r\,d\mathcal{H}^{n-1}\\
&= \frac{2}{r}\Big(\int_{B_1(0)}(\nabla\mathbf{u}_r\cdot\nabla(x\cdot\nabla\mathbf{u}_r-2\mathbf{u}_r)+\frac{\mathbf{u}_r}{|\mathbf{u}_r|}\cdot(x\cdot\nabla\mathbf{u}_r-2\mathbf{u}_r))\\
&\quad - 2\int_{\partial B_1(0)}\mathbf{u}_r\cdot(x\cdot\nabla\mathbf{u}_r-2\mathbf{u}_r)\,d\mathcal{H}^{n-1}\Big)\\
&= \frac{2}{r}\Big(\int_{B_1(0)}(-\Delta\mathbf{u}_r\cdot(x\cdot\nabla\mathbf{u}_r-2\mathbf{u}_r)+\frac{\mathbf{u}_r}{|\mathbf{u}_r|}\cdot(x\cdot\nabla\mathbf{u}_r-2\mathbf{u}_r))\\
&\quad + \int_{\partial B_1(0)}(x\cdot\nabla\mathbf{u}_r-2\mathbf{u}_r)\cdot(x\cdot\nabla\mathbf{u}_r-2\mathbf{u}_r)\,d\mathcal{H}^{n-1}\Big)\\
&= \frac{2}{r}\int_{\partial B_1(0)}|x\cdot\nabla\mathbf{u}_r-2\mathbf{u}_r|^2\,d\mathcal{H}^{n-1} = 2r\int_{\partial B_1(0)}\left|\frac{d\mathbf{u}_r}{dr}\right|^2\,d\mathcal{H}^{n-1}.
\end{aligned}$$

This proves the statement of the lemma. □

Note that for $x^0\in B_{1/2}$ and $r<1/2$,

$$W(\mathbf{u},x^0,r)\le C(\|\mathbf{u}\|_{W^{1,2}(B_1;\mathbb{R}^m)}+\|\mathbf{u}\|^2_{W^{1,2}(B_1;\mathbb{R}^m)}). \tag{18}$$

Moreover, we obtain the following properties:

Lemma 2. *1. The function $r\mapsto W(\mathbf{u},x^0,r)$ has a right limit $W(\mathbf{u},x^0,0+)\in[-\infty,+\infty)$ and in the case $D=\mathbb{R}^n$ it has also a limit $W(\mathbf{u},x^0,+\infty)\in(-\infty,+\infty]$.*

2. Let $0<r_k\to 0$ be a sequence such that the blow-up sequence

$$\mathbf{u}_k(x) := \frac{\mathbf{u}(x^0+r_kx)}{r_k^2}$$

converges weakly in $W^{1,2}_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^m)$ to $\mathbf{u}_0$. Then $\mathbf{u}_0$ is a homogeneous function of degree 2. Moreover

$$W(\mathbf{u},x^0,0+) = \int_{B_1(0)}|\mathbf{u}_0|\ge 0,$$

and $W(\mathbf{u},x^0,0+)=0$ implies that $u\equiv 0$ in $B_\delta(x^0)$ for some $\delta>0$.

3. The function $x\mapsto W(\mathbf{u},x,0+)$ is upper-semicontinuous.

Proof. 1. follows directly from the monotonicity formula.

2. By the assumption of convergence $(\mathbf{u}_k)_{k\in\mathbb{N}}$ is bounded in $W^{1,2}_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^m)$ and the limit $W(\mathbf{u},x^0,0+)$ is finite. From the monotonicity formula we obtain for all $0<\rho<\sigma<+\infty$ that

$$\int_\rho^\sigma\frac{1}{r^{n+4}}\int_{\partial B_r(0)}|x\cdot\nabla\mathbf{u}_k(x)-2\mathbf{u}_k(x)|^2\,d\mathcal{H}^{n-1}\,dr\to 0, k\to\infty,$$

proving the homogeneity of $\mathbf{u}_0$.

We calculate, using the homogeneity of $\mathbf{u}_0$ as well as Remark 2,

$$W(\mathbf{u},x^0,0+) = \int_{B_1(0)} (|\nabla \mathbf{u}_0|^2 + 2|\mathbf{u}_0|) - 2\int_{\partial B_1(0)} |\mathbf{u}_0|^2 \, d\mathcal{H}^{n-1}$$
$$= \int_{B_1(0)} (-\mathbf{u}_0 \cdot \Delta \mathbf{u}_0 + 2|\mathbf{u}_0|) + \int_{\partial B_1(0)} (x\cdot \nabla \mathbf{u}_0 \cdot \mathbf{u}_0 - 2|\mathbf{u}_0|^2)\, d\mathcal{H}^{n-1} = \int_{B_1(0)} |\mathbf{u}_0| \geq 0.$$

In the case $W(\mathbf{u},x^0,0+) = 0$ we obtain a contradiction to the non-degeneracy Lemma 1 unless $\mathbf{u} \equiv 0$ in some ball $B_\delta(x^0)$.

3. For $\epsilon > 0$, $M < +\infty$ and $x \in D$ we obtain from the monotonicity formula that

$$W(\mathbf{u},x,0+) \leq W(\mathbf{u},x,\rho) \leq \frac{\epsilon}{2} + W(\mathbf{u},x^0,\rho) \leq \begin{cases} \epsilon + W(\mathbf{u},x^0,0+), & W(\mathbf{u},x^0,0+) > -\infty, \\ -M, & W(\mathbf{u},x^0,0+) = -\infty, \end{cases}$$

if we choose first ρ and then $|x-x^0|$ small enough. □

5. A quadratic growth estimate

Theorem 2. *Any solution* $\mathbf{u}$ *to the system (1) in* $B_1(0)$ *satisfies*

$$|\mathbf{u}(x)| \leq C\mathbf{dist}^2(x,\Gamma_0(\mathbf{u})) \text{ and } |\nabla \mathbf{u}(x)| \leq C\mathbf{dist}(x,\Gamma_0(\mathbf{u})) \text{ for every } x \in B_{1/2}(0),$$

where the **dist** *denotes the Euclidean distance in* $\mathbb{R}^n$ *and constant C depends only on n and*

$$E(\mathbf{u},0,1) := \int_{B_1(0)} (|\nabla \mathbf{u}|^2 + 2|\mathbf{u}|).$$

Proof. The statement of the theorem is equivalent to

$$\sup_{B_r(x^0)} |\mathbf{u}| \leq C_1 r^2 \text{ and } \sup_{B_r(x^0)} |\nabla \mathbf{u}| \leq C_1 r \text{ for every } x^0 \in \Gamma_0(\mathbf{u}) \cap B_{1/2}(0) \text{ and every } r \in (0,1/4),$$

which in turn can be readily derived by standard elliptic theory from

$$\frac{1}{r^n}\int_{B_r(x^0)} |\mathbf{u}| \leq C_2 r^2 \text{ for all } x^0 \text{ and } r \text{ as above.} \tag{19}$$

Thus, our goal here is to show that (19) holds. To that end, notice first that by the monotonicity formula,

$$W(\mathbf{u},x^0,r) \leq W(\mathbf{u},x^0,1/2) \leq 2^{n+2}E(\mathbf{u},0,1) \text{ for every } x^0 \in B_{1/2}(0)\cap\Gamma_0(\mathbf{u}) \text{ and } r \leq 1/2.$$

Therefore

$$\frac{2}{r^{n+2}}\int_{B_r(x^0)} |\mathbf{u}| = W(\mathbf{u},x^0,r) - \frac{1}{r^{n+2}}\int_{B_r(x^0)} |\nabla \mathbf{u}|^2 + \frac{2}{r^{n+3}}\int_{\partial B_r(x^0)} |\mathbf{u}|^2\, d\mathcal{H}^{n-1}$$
$$= W(\mathbf{u},x^0,r) - \frac{1}{r^{n+2}}\int_{B_r(x^0)} |\nabla(\mathbf{u} - S_{x^0}\mathbf{p})|^2 + \frac{2}{r^{n+3}}\int_{\partial B_r(x^0)} |\mathbf{u} - S_{x^0}\mathbf{p}|^2\, d\mathcal{H}^{n-1}$$
$$\leq 2^{n+2}E(\mathbf{u},0,1) + \frac{2}{r^{n+3}}\int_{\partial B_r(x^0)} |\mathbf{u} - S_{x^0}\mathbf{p}|^2\, d\mathcal{H}^{n-1}$$

for each $\mathbf{p} = (p_1,\dots,p_m) \in \mathcal{H}$; here the set $\mathcal{H}$ is the set of all $\mathbf{p} = (p_1,\dots,p_m)$ such that each component p_j is a homogeneous harmonic polynomial of second order, $S_{x^0}\mathbf{f}(x) := \mathbf{f}(x-x^0)$.

Let $x^0 \in \Gamma_0$ and $\mathbf{p}_{x^0,r}$ be the minimiser of $\int_{\partial B_r(x^0)} |\mathbf{u} - S_{x^0}\mathbf{p}|^2\, d\mathcal{H}^{n-1}$ in $\mathcal{H}$. It follows that

$$0 = \int_{\partial B_r(x^0)} (\mathbf{u} - S_{x^0}\mathbf{p}_{x^0,r})\cdot S_{x^0}\mathbf{q}\, d\mathcal{H}^{n-1} \text{ for every } \mathbf{q} \in \mathcal{H}. \tag{20}$$

We maintain that there is a constant C_1 depending only on the dimension n as well as $E(\mathbf{u},0,1)$ such that for each $x^0 \in B_{1/2}(0)\cap\Gamma_0$ and $r \le 1/4$,

$$\frac{1}{r^{n+3}}\int_{\partial B_r(x^0)} |\mathbf{u} - S_{x^0}\mathbf{p}_{x^0,r}|^2 \, d\mathcal{H}^{n-1} \le C_1.$$

Suppose towards a contradiction that there is a sequence of solutions $\mathbf{u}_k$ (to equation (1) in $B_1(0)$) and a sequence of points $x^k \in B_{1/2}(0)\cap\Gamma_0(\mathbf{u}_k)$ as well as $r_k \to 0$ such that $I(\mathbf{u}_k,0,1)$ are uniformly bounded,

$$M_k := \frac{1}{r_k^{n+3}}\int_{\partial B_{r_k}(x^k)} |\mathbf{u}_k - S_{x^k}\mathbf{p}_{x^k,r_k}|^2 \, d\mathcal{H}^{n-1} \to \infty\,.$$

For $\mathbf{v}_k(x) := \mathbf{u}_k(x^k + r_k x)/r_k^2$, and $\mathbf{w}_k(x) := \left(\mathbf{v}_k - \mathbf{p}_{x^k,r_k}\right)/M_k$, we have $\|\mathbf{w}_k\|_{L^2(\partial B_1(0);\mathbb{R}^m)} = 1$ and

$$\begin{aligned}\int_{B_1(0)} |\nabla \mathbf{w}_k|^2 - 2\int_{\partial B_1(0)} |\mathbf{w}_k|^2 &= M_k^{-2}\left(\int_{B_1(0)} |\nabla(\mathbf{v}_k - \mathbf{p}_{x^k,r_k})|^2 - 2\int_{\partial B_1(0)} |\mathbf{v}_k - \mathbf{p}_{x^k,r_k}|^2 \, d\mathcal{H}^{n-1}\right)\\ &= M_k^{-2}\left(\int_{B_1(0)} |\nabla \mathbf{v}_k|^2 - 2\int_{\partial B_1(0)} |\mathbf{v}_k|^2 \, d\mathcal{H}^{n-1}\right)\\ &\le M_k^{-2} W(\mathbf{u}_k, x^k, r_k) \le M_k^{-2} C_2 \to 0,\ k \to \infty.\end{aligned}$$

It follows that $(\mathbf{w}_k)_{k\in\mathbb{N}}$ is bounded in $W^{1,2}(B_1;\mathbb{R}^m)$ such that —passing to a subsequence if necessary— $\mathbf{w}_k$ converges weakly in $W^{1,2}(B_1;\mathbb{R}^m)$ to $\mathbf{w}_0 \in W^{1,2}(B_1;\mathbb{R}^m)$. By Rellich's theorem, $\mathbf{w}_k$ converges strongly in $L^2(\partial B_1(0);\mathbb{R}^m)$, $\|\mathbf{w}_0\|_{L^2(\partial B_1(0);\mathbb{R}^m)} = 1$ and (by (20)) $0 = \int_{\partial B_1(0)} \mathbf{w}_0 \cdot \mathbf{q} \, d\mathcal{H}^{n-1}$ for every $\mathbf{q} \in \mathcal{H}$. Hence we obtain that

$$\int_{B_1(0)} |\nabla \mathbf{w}_0|^2 \le 2\int_{\partial B_1(0)} |\mathbf{w}_0|^2 \, d\mathcal{H}^{n-1} = 2. \tag{21}$$

Moreover,

$$|\Delta \mathbf{w}_k| \le \frac{C_3}{M_k} \to 0, k \to \infty,$$

such that $\mathbf{w}_0$ is harmonic in $B_1(0)$ and (by $C^{1,\alpha}$-estimates) $|\mathbf{w}_0|(0) = |\nabla \mathbf{w}_0|(0) = 0$. Now by [17, Lemma 4.1], each component z_j of $\mathbf{w}_0$ must satisfy

$$2\int_{\partial B_1(0)} z_j^2 \, d\mathcal{H}^{n-1} \le \int_{B_1(0)} |\nabla z_j|^2.$$

Summing over j we obtain

$$2\int_{\partial B_1(0)} |\mathbf{w}_0|^2 \, d\mathcal{H}^{n-1} \le \int_{B_1(0)} |\nabla \mathbf{w}_0|^2,$$

implying by (21) that

$$2\int_{\partial B_1(0)} |\mathbf{w}_0|^2 \, d\mathcal{H}^{n-1} = \int_{B_1(0)} |\nabla \mathbf{w}_0|^2.$$

Thus

$$2\int_{\partial B_1(0)} z_j^2 \, d\mathcal{H}^{n-1} = \int_{B_1(0)} |\nabla z_j|^2$$

for each j, implying by [17, Lemma 4.1] that z_j is a homogeneous harmonic polynomial of second order. But then $0 = \int_{\partial B_1(0)} \mathbf{w}_0 \cdot \mathbf{q} \, d\mathcal{H}^{n-1}$ for every $\mathbf{q} \in \mathcal{H}$ implies that $\mathbf{w}_0 = 0$ on $\partial B_1(0)$, contradicting $\|\mathbf{w}_0\|_{L^2(\partial B_1;\mathbb{R}^m)} = 1$. □

The next section follows closely the procedure in [19] and [16].

6. An energy decay estimate and uniqueness of blow-up limits

In this section we show that an epiperimetric inequality always implies an energy decay estimate and uniqueness of blow-up limits. More precisely:

Theorem 3 (Energy decay and uniqueness of blow-up limits)**.** *Let $x^0 \in \Gamma_0(\mathbf{u})$, and suppose that the epiperimetric inequality holds with $\kappa \in (0,1)$ for each*

$$\mathbf{c}_r(x) := |x|^2 \mathbf{u}_r(\frac{x}{|x|}) = \frac{|x|^2}{r^2}\mathbf{u}(x^0 + \frac{r}{|x|}x)$$

and for all $r \le r_0 < 1$. Finally let $\mathbf{u}_0$ denote an arbitrary blow-up limit of $\mathbf{u}$ at x^0. Then

$$\left|W(\mathbf{u}, x^0, r) - W(\mathbf{u}, x^0, 0+)\right| \;\le\; \left|W(\mathbf{u}, x^0, r_0) - W(\mathbf{u}, x^0, 0+)\right| \left(\frac{r}{r_0}\right)^{\frac{(n+2)\kappa}{1-\kappa}}$$

for $r \in (0, r_0)$, and there exists a constant C depending only on n and κ such that

$$\int_{\partial B_1(0)} \left|\frac{\mathbf{u}(x^0 + rx)}{r^2} \;-\; \mathbf{u}_0(x)\right| \, d\mathcal{H}^{n-1} \;\le\; C\, \left|W(\mathbf{u}, x^0, r_0) - W(\mathbf{u}, x^0, 0+)\right|^{\frac12} \left(\frac{r}{r_0}\right)^{\frac{(n+2)\kappa}{2(1-\kappa)}}$$

for $r \in (0, \frac{r_0}{2})$, and $\mathbf{u}_0$ is the unique blow-up limit of $\mathbf{u}$ at x^0 .

Proof. We define

$$e(r) := r^{-n-2}\int_{B_r(x^0)} (|\nabla \mathbf{u}|^2 \;+\; 2|\mathbf{u}|) \;-\; 2\, r^{-n-3}\int_{\partial B_r(x^0)} |\mathbf{u}|^2\, d\mathcal{H}^{n-1} \;-\; W(\mathbf{u}, x^0, 0+).$$

Up to a constant $e(r)$ is the function of the monotonicity identity, so that we have already computed $e'(r)$. Here however, we need a different formula for $e'(r)$:

$$\begin{aligned}
e'(r) \;=\; \Bigg[-\frac{n+2}{r}e(r) \;-\; \frac{n+2}{r}W(\mathbf{u}, x^0, 0+) \;+\; \frac{2}{r}\, r^{-n-3}\int_{\partial B_r(x^0)} |\mathbf{u}|^2\, d\mathcal{H}^{n-1} \\
-\, 2\, r^{-n-3}\int_{\partial B_r(x^0)} 2\nu\cdot\nabla\mathbf{u}\cdot\mathbf{u}\, d\mathcal{H}^{n-1} \;-\; \frac{2(n-1)}{r}\, r^{-n-3}\int_{\partial B_r(x^0)} |\mathbf{u}|^2\, d\mathcal{H}^{n-1} \\
+\, r^{-n-2}\int_{\partial B_r(x^0)} (|\nabla \mathbf{u}|^2 \;+\; 2|\mathbf{u}|)\, d\mathcal{H}^{n-1}\Bigg] \;=\; r^{-1}\Bigg[\int_{\partial B_1(0)} (|\nabla \mathbf{u}_r|^2 \\
+\, 2|\mathbf{u}_r| \;-\; 4\nu\cdot\nabla\mathbf{u}_r\cdot\mathbf{u}_r \;+\; 4|\mathbf{u}_r|^2 \;+\; 4|\mathbf{u}_r|^2 \;-\; 2(n+2)|\mathbf{u}_r|^2)\, d\mathcal{H}^{n-1} \;-\; (n+2)W(\mathbf{u}, x^0, 0+)\Bigg] \\
-\, \frac{n+2}{r}e(r) \;\ge\; r^{-1}\Bigg[\int_{\partial B_1(0)} (|\nabla_\theta \mathbf{u}_r|^2 \;+\; 2|\mathbf{u}_r| \;+\; 4|\mathbf{u}_r|^2 \;-\; 2(n+2)|\mathbf{u}_r|^2)\, d\mathcal{H}^{n-1} \\
-\, (n+2)W(\mathbf{u}, x^0, 0+)\Bigg] \;-\; \frac{n+2}{r}e(r) \;=\; r^{-1}\Bigg[\int_{\partial B_1(0)} (|\nabla_\theta \mathbf{c}_r|^2 \;+\; 2|\mathbf{c}_r| \;+\; |\nu\cdot\nabla\mathbf{c}_r|^2)\, d\mathcal{H}^{n-1} \\
-\, (n+2)2\int_{\partial B_1(0)} |\mathbf{c}_r|^2\, d\mathcal{H}^{n-1} \;-\; (n+2)W(\mathbf{u}, x^0, 0+)\Bigg] \;-\; \frac{n+2}{r}e(r) \\
=\; \frac{n+2}{r}\Big[M(\mathbf{c}_r) - W(\mathbf{u}, x^0, 0+) - e(r)\Big] \;.
\end{aligned}$$

Here we employ the minimality of $\mathbf{u}$ as well as the assumption that the epiperimetric inequality $M(\mathbf{v}) \le (1-\kappa)M(\mathbf{c}_r) \;+\; \kappa\, W(\mathbf{u}, x^0, 0+)$ holds for some $\mathbf{v} \in W^{1,2}(B_1; \mathbb{R}^m)$ with $\mathbf{c}_r$-boundary values and we obtain for $r \in (0, r_0)$ the estimate

$$e'(r) \ge \frac{n+2}{r}\frac{1}{1-\kappa}(M(\mathbf{u}_r) - W(\mathbf{u}, x^0, 0+)) \;-\; \frac{n+2}{r}e(r)$$

$$= \frac{n+2}{r}\left(\frac{1}{1-\kappa}-1\right)e(r) = \frac{(n+2)\kappa}{1-\kappa}\frac{1}{r}\, e(r)\ .$$

By the monotonicity formula Lemma 1, $e(r) \geq 0$, and we conclude in the non-trivial case $e > 0$ in (r_1, r_0) that

$$(\log(e(s)))' \ \geq \ \frac{(n+2)\kappa}{1-\kappa}\frac{1}{s} \quad \text{for } s \in (r_1, r_0)\ .$$

Integrating from r to r_0 we obtain that

$$\log\left(\frac{e(r_0)}{e(r)}\right) \ \geq \ \frac{(n+2)\kappa}{1-\kappa}\log\left(\frac{r_0}{r}\right) \ \text{ and } \ \frac{e(r_0)}{e(r)} \ \geq \ \left(\frac{r_0}{r}\right)^{\frac{(n+2)\kappa}{1-\kappa}} \quad \text{for } r \in (r_1, r_0)$$

and that $e(r) \leq e(r_0)\ \left(\frac{r}{r_0}\right)^{\frac{(n+2)\kappa}{1-\kappa}}$ for $r \in (0, r_0)$ which proves our first statement.
Using once more the monotonicity formula (Lemma 1) we get for $0 < \rho < \sigma \leq r_0$ an estimate of the form

$$\int_{\partial B_1(0)}\int_\rho^\sigma \left|\frac{d\mathbf{u}_r}{dr}\right| dr\, d\mathcal{H}^{n-1} \leq \int_\rho^\sigma r^{-1-n}\int_{\partial B_r(x^0)}\left|\nu\cdot\nabla\mathbf{u} \ - \ 2\frac{\mathbf{u}}{r}\right|\, d\mathcal{H}^{n-1}\, dr$$

$$\leq \ \sqrt{n\,\omega_n}\int_\rho^\sigma r^{-1-n} r^{\frac{n-1}{2}} r^{\frac{n+2}{2}}\left(r^{-n-2}\int_{\partial B_r(x^0)}\left|\nu\cdot\nabla\mathbf{u} \ - \ 2\frac{\mathbf{u}}{r}\right|^2 d\mathcal{H}^{n-1}\right)^{\frac{1}{2}} dr$$

$$= \ \sqrt{\frac{n\,\omega_n}{2}}\int_\rho^\sigma r^{-\frac{1}{2}}\ \sqrt{e'(r)}\, dr \ \leq \ \sqrt{\frac{n\,\omega_n}{2}}(\log(\sigma)-\log(\rho))^{\frac{1}{2}}\ (e(\sigma)-e(\rho))^{\frac{1}{2}}\ .$$

Considering now $0 < 2\rho < 2r \leq r_0$ and intervals $[2^{-k-1}, 2^{-k}) \ni \rho$ and $[2^{-\ell-1}, 2^{-\ell}) \ni r$ the already proved part of the theorem yields that

$$\int_{\partial B_1(0)}\left|\frac{\mathbf{u}(x^0+rx)}{r^2} \ - \ \frac{\mathbf{u}(x^0+\rho x)}{\rho^2}\right| d\mathcal{H}^{n-1} \leq \sum_{i=\ell}^{k}\int_{\partial B_1(0)}\int_{2^{-i-1}}^{2^{-i}}\left|\frac{d\mathbf{u}_r}{dr}\right| dr\, d\mathcal{H}^{n-1}$$

$$\leq \ C_1(n)\sum_{i=\ell}^{k}(\log(2^{-i})-\log(2^{-i-1}))^{\frac{1}{2}}(e(2^{-i})-e(2^{-i-1}))^{\frac{1}{2}} \ = \ C_2(n)\sum_{i=\ell}^{k}(e(2^{-i})-e(2^{-i-1}))^{\frac{1}{2}}$$

$$\leq \ C_3(n,\kappa)\left|W(\mathbf{u},x^0,r_0)-W(\mathbf{u},x^0,0+)\right|^{\frac{1}{2}}\sum_{j=\ell}^{+\infty}\left(r_0\, 2^j\right)^{\frac{-(n+2)\kappa}{2(1-\kappa)}}$$

$$\leq \ C_4(n,\kappa)\left|W(\mathbf{u},x^0,r_0)-W(\mathbf{u},x^0,0+)\right|^{\frac{1}{2}}\ \frac{c^\ell}{1-c}\ r_0^{\frac{-(n+2)\kappa}{2(1-\kappa)}}$$

where $c = 2^{\frac{-(n+2)\kappa}{2(1-\kappa)}} \in (0,1)$. Thus

$$\int_{\partial B_1(0)}\left|\frac{\mathbf{u}(x^0+rx)}{r^2} \ - \ \frac{\mathbf{u}(x^0+\rho x)}{\rho^2}\right| d\mathcal{H}^{n-1}$$

$$\leq \ C_5(n,\kappa)\left|W(\mathbf{u},x^0,r_0)-W(\mathbf{u},x^0,0+)\right|^{\frac{1}{2}}\left(\frac{r}{r_0}\right)^{\frac{(n+2)\kappa}{2(1-\kappa)}}\ ,$$

and letting $\frac{\mathbf{u}(x^0+\rho_j x)}{\rho_j{}^2} \to \mathbf{u}_0$ as a certain sequence $\rho_j \to 0$ finishes our proof. □

7. Homogeneous solutions

In this section we consider homogeneous solutions $\mathbf{u} \in W^{1,2}(B_1;\mathbb{R}^m)$, meaning that

$$\mathbf{u}(\lambda x) = \lambda^2 \mathbf{u}(x) \quad \text{for all } \lambda > 0 \text{ and } x \in B_1(0).$$

Obviously $\mathbf{u}$ may be extended to a homogeneous solution on $\mathbb{R}^n$.

Moreover, if $\mathbf{dist}_{L^1(B_1;\mathbb{R}^m)}(\mathbf{u}, \mathbb{H}) \le 1$ (see (4)) then due to Remark 1 we have

$$\sup_{B_1} |\mathbf{u}| \le C(n,m) \text{ and } \sup_{B_1} |\nabla \mathbf{u}| \le C(n,m). \tag{22}$$

Proposition 3. *If $B_1 \cap \mathbf{supp}\, \mathbf{u} \subset \{x_n > -\delta(n, \sup_{B_1(0)} |\mathbf{u}|)\}$, then $\mathbf{u} \in \mathbb{H}$.*

Proof. Observe first that each component u_i is a solution of

$$\mathcal{L}_i u_i := -\Delta' u_i + u_i/|\mathbf{u}| = 2n u_i$$

in every connected component Ω' of $\partial B_1 \cap \{|\mathbf{u}| > 0\}$, where Δ' is the Laplace-Beltrami operator on the unit sphere in $\mathbb{R}^n$. In Lemma 4 of the Appendix we prove that $u_i = a_i f_{\Omega'}$ for a real number a_i and a function $f_{\Omega'}$ depending only on Ω' which is positive on Ω' and vanishes on the boundary of Ω'. It follows that for each connected component Ω of $B_1 \cap \{|\mathbf{u}| > 0\}$ there exists a unit vector $\mathbf{a} = (a_1 \dots, a_m)$ such that $\mathbf{u}(x) = \mathbf{a}|\mathbf{u}(x)|$ and $\Delta|\mathbf{u}| = 1$ in Ω.

Now, if $|\nabla \mathbf{u}| = 0$ on $\partial\Omega$, then we may extend $\mathbf{u}$ by 0 outside Ω, that is $|\mathbf{u}|$ can be extended to a 2-homogeneous non-negative solution of the classical obstacle problem in $\mathbb{R}^n$. These solutions have been completely classified (see [5], cf. also [13]), and $\mathbf{supp}\, \mathbf{u} \subset \{x_n > -\delta|x|\}$ (where $\delta = \delta(n, m, \sup_{B_1(0)} |\mathbf{u}|)$) would in this case imply that up to rotation, $|\mathbf{u}| = h$, and $\mathbf{u} = \mathbf{a}h$, where h is a half-space solution for scalar problem.

If, on the other hand, there is a point $x^0 \in \partial\Omega \cap \{|\nabla \mathbf{u}| \neq 0\}$, then the fact that $\mathbf{u}$ is continuously differentiable, implies that $\mathbf{a}$ equals the vector of the adjoining connected component of $\{|\mathbf{u}| > 0\}$ up to the sign. In this case we obtain, taking the maximal union of all such connected components, that each u_i is a 2-homogeneous solution of the scalar two-phase obstacle problem

$$\Delta v = c(\chi_{\{v>0\}} - \chi_{\{v<0\}}) \text{ in } \mathbb{R}^n$$

with $c > 0$, satisfying $v = 0$ in $\{x_n \le -\delta\}$. However, according to [14, Theorem 4.3], no such solution exists. ◻

Lemma 3. *The half-plane solutions are (in the $L^1(B_1(0);\mathbb{R}^m)$-topology) isolated within the class of homogeneous solutions of degree* 2.

Proof. Let $\|\mathbf{u} - \mathbf{h}\|_{L^1(B_1;\mathbb{R}^m)} \le \epsilon$, where rotating in $\mathbb{R}^n$ and in $\mathbb{R}^m$ if necessary we may assume that

$$\mathbf{h}(x) = \frac{\max(x_n, 0)^2}{2} \mathbf{e}^1.$$

From (22) as well as Propositions 2 and 3 we infer that $\mathbf{u} \in \mathbb{H}$ if ϵ has been chosen small enough, depending only on n and m. ◻

We defined in (5) the constant $\alpha_n = 2M(\mathbf{h})$ where $\mathbf{h} \in \mathbb{H}$. Now we are going to estimate the value of $M(\mathbf{u})$ for an arbitrary homogeneous solution $\mathbf{u}$ of degree 2.

Proposition 4.

$$\alpha_n = \frac{\mathcal{H}^{n-1}(\partial B_1)}{2n(n+2)}. \tag{23}$$

Let $\mathbf{u}$ *be a homogeneous solution of degree* 2*. Then*

$$M(\mathbf{u}) \geq \alpha_n \frac{\mathcal{H}^{n-1}(\partial B_1 \cap \{|\mathbf{u}| > 0\})}{\mathcal{H}^{n-1}(\partial B_1)}. \tag{24}$$

In particular,

$$M(\mathbf{u}) \geq \alpha_n \textit{ if } |\mathbf{u}| > 0 \textit{ a.e.} \tag{25}$$

Proof. Let $U := |\mathbf{u}|$, and recall (17):

$$\Delta U = 1 + \frac{A}{U} \text{ in } \{|\mathbf{u}| > 0\}, \text{ where } A = |\nabla \mathbf{u}|^2 - |\nabla U|^2 \geq 0.$$

It follows that —using the homogeneity of $\mathbf{u}$—

$$\begin{aligned}
\int_{B_1 \cap \{|\mathbf{u}|>0\}} \left(1 + \frac{A}{U}\right) &= \int_{\partial(B_1 \cap \{|\mathbf{u}|>0\})} \nabla U \cdot \nu \, d\mathcal{H}^{n-1} \\
&= 2 \int_{\partial B_1 \cap \{|\mathbf{u}|>0\}} U \, d\mathcal{H}^{n-1} - 2 \int_{B_1 \cap \{\mathbf{u}=0\} \cap \{|\nabla \mathbf{u}|>0\}} |\nabla U| \, d\mathcal{H}^{n-1} \\
&= 2(n+2) \int_{B_1} U - 2 \int_{B_1 \cap \{\mathbf{u}=0\} \cap \{|\nabla \mathbf{u}|>0\}} |\nabla U| \, d\mathcal{H}^{n-1}.
\end{aligned}$$

On the other hand, using once more the homogeneity of $\mathbf{u}$,

$$M(\mathbf{u}) = \int_{B_1} (|\nabla \mathbf{u}|^2 + 2|\mathbf{u}|) - 2 \int_{\partial B_1} |\mathbf{u}|^2 \, d\mathcal{H}^{n-1} = \int_{B_1} |\mathbf{u}| = \int_{B_1} U. \tag{26}$$

In order to verify (23), observe that for $\mathbf{e} \in \partial B_1 \subset \mathbb{R}^m$ and $\mathbf{h}(x) = \mathbf{e} \max(x_n, 0)^2/2$,

$$\frac{\alpha_n}{2} = M(\mathbf{h}) = \frac{1}{2} \int_{B_1} \max(x_n, 0)^2 = \frac{1}{4} \int_{B_1} x_n^2 = \frac{1}{4n} \int_{B_1} |x|^2 = \frac{\mathcal{H}^{n-1}(\partial B_1)}{4n(n+2)}.$$

Using the above estimates we conclude that

$$M(\mathbf{u}) \geq \frac{1}{2(n+2)} |B_1 \cap \{|\mathbf{u}| > 0\}| = \frac{\mathcal{H}^{n-1}(\partial B_1 \cap \{|\mathbf{u}| > 0\})}{2n(n+2)} = \alpha_n \frac{\mathcal{H}^{n-1}(\partial B_1 \cap \{|\mathbf{u}| > 0\})}{\mathcal{H}^{n-1}(\partial B_1)}.$$

□

Corollary 1. *Let* $\mathbf{u}$ *be a homogeneous solution of degree* 2*. Then*

$$M(\mathbf{u}) \geq \alpha_n \max\left(\frac{1}{2}, \frac{\mathcal{H}^{n-1}(\partial B_1 \cap \{|\mathbf{u}| > 0\})}{\mathcal{H}^{n-1}(\partial B_1)}\right) \geq \alpha_n/2, \tag{27}$$

and $M(\mathbf{u}) = \alpha_n/2$ *implies that* $\mathbf{u} \in \mathbb{H}$*. Moreover,* $\alpha_n/2 < \bar{\alpha}_n := \inf\{M(\mathbf{v}) : \mathbf{v}$ *is a homogeneous solution of degree* 2*, but* $\mathbf{v} \notin \mathbb{H}\}$*.*

Proof. If $\mathcal{H}^{n-1}(\partial B_1 \cap \{|\mathbf{u}| = 0\}) = 0$, then (27) follows from (25). Otherwise $\{|\mathbf{u}| = 0\}$ contains by the non-degeneracy property Lemma 1 an open ball $B_\rho(y)$, and we may choose it in such a way that there is a point $z \in \partial B_\rho(y) \cap \partial\{|\mathbf{u}| > 0\}$. Let $\mathbf{u}_0$ be a blow up of $\mathbf{u}$ at z. Since $\textbf{supp } \mathbf{u}_0$ is contained in a half-space it follows from Proposition 3 that $\mathbf{u}_0 \in \mathbb{H}$. Note that by homogeneity, $|\mathbf{u}(x)| \leq C|x|^2$ and $|\nabla \mathbf{u}(x)| \leq C|x|$ for every $x \in \mathbb{R}^n$, implying that the limit $W(\mathbf{u}, x^0, +\infty)$ does not depend on the choice of x^0.

From (23) we obtain therefore that

$$\frac{\alpha_n}{2} = M(\mathbf{u}_0) = W(\mathbf{u}, z, 0^+) \leq W(\mathbf{u}, z, +\infty) = W(\mathbf{u}, 0, +\infty) = M(\mathbf{u}). \tag{28}$$

Now we have to prove that $M(\mathbf{u}) = \alpha_n/2$ implies $\mathbf{u} \in \mathbb{H}$. Consider a ball $B_\rho(y)$ and a point z as above, that is $y = z + \rho e$ with a unit vector e. It follows from homogeneity of $\mathbf{u}$ that e is orthogonal to z. We consider two cases.

Case a) If $z = 0$ then, again due to homogeneity of $\mathbf{u}$, we have $|\mathbf{u}(x)| = 0$ in a half-space $(x \cdot e) > 0$. Hence $\mathbf{u} \in \mathbb{H}$ by Proposition 3.
Case b) If $|z| > 0$ then, since $W(\mathbf{u}, z, r)$ does not depend on r (by (28)), we conclude that u is homogeneous with homogeneity center z. More exactly, we have $\mathbf{u}(z + kx) = k^2\mathbf{u}(z + x)$ for any $k > 0, x \in \mathbb{R}^n$. Since also $\mathbf{u}(z + kx) = k^2\mathbf{u}(z/k + x)$ we obtain $u(z + x) = u(z/k + x)$ for any $k > 0, x \in \mathbb{R}^n$. It means that $\mathbf{u}$ is constant in direction of vector z. In particular, $|\mathbf{u}| = 0$ in the ball $B_\rho(\rho e)$ touching the origin and we are again at the case a).

Last, we have to prove $\bar{\alpha}_n > \alpha_n/2$. If it is not true then there is a sequence of homogeneous global solutions $\{\mathbf{u}_k\}$ such that

$$M(\mathbf{u}_k) \searrow \frac{\alpha_n}{2} \quad \text{as} \quad k \to \infty.$$

In particular it implies by (26) uniform boundedness of $\mathbf{u}_k$ in $L^1(B_1(0))$ and therefore, by (16) and by elliptic theory, uniform boundedness of solutions $\mathbf{u}_k$ in $W^{2,q}_{\mathrm{loc}}(\mathbb{R}^n)$ for any $q < \infty$. Then there exists a limit $\tilde{\mathbf{u}}$, by subsequence, such that $\tilde{\mathbf{u}}$ is a homogeneous solution, $\tilde{\mathbf{u}} \notin \mathbb{H}$ (by Lemma 3) and $M(\tilde{\mathbf{u}}) = \alpha_n/2$. From the first part of the proof we infer that $\tilde{\mathbf{u}} \in \mathbb{H}$, and a contradiction arises. □

Definition 1. *A point x is a regular free boundary point for $\mathbf{u}$ if:*

$$x \in \Gamma_0(\mathbf{u}) \qquad \text{and} \qquad \lim_{r \to 0} W(\mathbf{u}, x, r) = \frac{\alpha_n}{2}.$$

We denote by $\mathcal{R}_\mathbf{u}$ the set of all regular free boundary points of $\mathbf{u}$ in B_1.

Corollary 2. *The set of regular free boundary points $\mathcal{R}_\mathbf{u}$ is open relative to $\Gamma_0(\mathbf{u})$.*

Proof. This is an immediate consequence of Corollary 1 and the upper semicontinuity Lemma 2. □

8. Regularity

In this last section we prove that the set of regular free boundary points $\mathcal{R}_\mathbf{u}$ is locally in D a $C^{1,\beta}$-surface and we derive a *macroscopic criterion for regularity:* suppose that $W(\mathbf{u}, x, r)$ drops for some (not necessarily small) r below the critical value $\bar{\alpha}_n$: then $\partial\{|\mathbf{u}| > 0\}$ must be a $C^{1,\beta}$-surface in an open neighborhood of x .

Theorem 4. *Let C_h be a compact set of points $x^0 \in \Gamma_0(\mathbf{u})$ with the following property: at least one blow-up limit $\mathbf{u}_0$ of $\mathbf{u}$ at x^0 is a half-plane solution, say $\mathbf{u}_0(x) = \frac{1}{2}\mathbf{e}\max(x \cdot \nu(x^0), 0)^2$ for some $\nu(x^0) \in \partial B_1(0) \subset \mathbb{R}^n$ and $\mathbf{e}(x^0) \in \partial B_1(0) \subset \mathbb{R}^m$. Then there exist $r_0 > 0$ and $C < \infty$ such that*

$$\int_{\partial B_1(0)} \left| \frac{\mathbf{u}(x^0 + rx)}{r^2} - \frac{1}{2}\mathbf{e}(x^0)\max(x \cdot \nu(x^0), 0)^2 \right| d\mathcal{H}^{n-1} \leq C\, r^{\frac{(n+2)\kappa}{2(1-\kappa)}}$$

for every $x^0 \in C_h$ and every $r \leq r_0$.

Proof. **Step 1:** Due to Dini's theorem, $W(\mathbf{u}, x, r) \leq \epsilon + \alpha_n/2$ for all $r \in (0, r_0)$ and all $x \in C_h$.

Step 2: If $\rho_j \to 0, x^j \in C_h$ and $\mathbf{u}_j := \mathbf{u}(x^j + \rho_j \cdot)/\rho_j^2 \to \mathbf{v}$ in $W^{1,2}_{\mathrm{loc}}(\mathbb{R}^n; \mathbb{R}^m)$ as $j \to \infty$, then $\mathbf{v} \in \mathbb{H}$: According to Remark 2, $\mathbf{v}$ is a global solution of (1). Moreover, by Step 1

$$W(\mathbf{v}, 0, \rho) = \lim_{j \to \infty} W(\mathbf{u}_j, 0, \rho) = \lim_{j \to \infty} W(\mathbf{u}, x^j, \rho_j\rho) = \frac{\alpha_n}{2}.$$

But then Lemma 1 and Corollary 1 imply that $\mathbf{v} \in \mathbb{H}$.

Step 3: For small ρ, $\mathbf{u}(x+\rho\cdot)/\rho^2$ is uniformly (in $x \in C_h$) close to $\mathbb{H}$ in the $W^{1,2}_{\rm loc}(\mathbb{R}^n;\mathbb{R}^m)\cap L^\infty_{\rm loc}(\mathbb{R}^n;\mathbb{R}^m)$-topology: assuming towards a contradiction that this is not the case, we obtain $\rho_j \to 0$ and $x^j \in C_h$ such that the distance in the same topology is $\geq \delta > 0$. As $\mathbf{u}_j := \mathbf{u}(x^j + \rho_j\cdot)/\rho_j^2$ is by Theorem 2 and $W^{2,p}$-theory bounded in $W^{2,p}_{\rm loc}(\mathbb{R}^n;\mathbb{R}^m)$ for each $q \in [1,+\infty)$, passing to a subsequence if necessary $\mathbf{u}_j \to \mathbf{w}$ in $W^{1,2}_{\rm loc}(\mathbb{R}^n;\mathbb{R}^m)\cap C^0_{\rm loc}(\mathbb{R}^n;\mathbb{R}^m)$, implying by Step 2 that $\mathbf{w} \in \mathbb{H}$, contradicting the distance being $\geq \delta > 0$. □

Theorem 5 (regularity)**.** *The free boundary $\partial\{|\mathbf{u}| > 0\}$ is in an open neighborhood of the set $\mathcal{R}_{\mathbf{u}}$ locally a $C^{1,\beta}$-surface; here $\beta = \frac{(n+2)\kappa}{2(1-\kappa)}\left(1 + \frac{(n+2)\kappa}{2(1-\kappa)}\right)^{-1}$.*

Proof. Let us consider a point $x^0 \in \mathcal{R}_{\mathbf{u}}$. By Theorem 4 there exists $\delta_0 > 0$ such that $B_{2\delta_0}(x^0) \subset D$ and

$$\int_{\partial B_1(0)} \left| \frac{\mathbf{u}(x^1 + rx)}{r^2} - \frac{1}{2}\mathbf{e}(x^1)\max(x\cdot\nu(x^1),0)^2 \right| d\mathcal{H}^{n-1} \leq C\, r^{\frac{(n+2)\kappa}{2(1-\kappa)}} \tag{29}$$

$$\text{for every } x^1 \in \mathcal{R}_{\mathbf{u}} \cap \overline{B_{\delta_0}(x^0)} \text{ and for every } r \leq \min(\delta_0, r_0)\ .$$

We now observe that $x^1 \mapsto \nu(x^1)$ and $x^1 \mapsto \mathbf{e}(x^1)$ are Hölder-continuous with exponent β on $\mathcal{R}_{\mathbf{u}} \cap \overline{B_{\delta_1}(x^0)}$ for some $\delta_1 \in (0,\delta_0)$:

$$\frac{1}{2}\int_{\partial B_1(0)} \left|\mathbf{e}(x^1)\max(x\cdot\nu(x^1),0)^2 - \mathbf{e}(x^2)\max(x\cdot\nu(x^2),0)^2\right| d\mathcal{H}^{n-1} \leq 2C\, r^{\frac{(n+2)\kappa}{2(1-\kappa)}}$$

$$+ \int_{\partial B_1(0)}\int_0^1 \left|\frac{\nabla\mathbf{u}(x^1 + rx + t(x^2 - x^1))}{r^2}\right| |x^1 - x^2|\, dt\, d\mathcal{H}^{n-1} \leq 2C\, r^{\frac{(n+2)\kappa}{2(1-\kappa)}}$$

$$+ C_1\, \frac{\max(r, |x^1 - x^2|)|x^1 - x^2|}{r^2} \leq (2C + C_1)|x^2 - x^1|^{\gamma\frac{(n+2)\kappa}{2(1-\kappa)}}$$

if we choose $\gamma := \left(1 + \frac{(n+2)\kappa}{2(1-\kappa)}\right)^{-1}$ and $r := |x^2 - x^1|^\gamma \leq \min(\delta_0, r_0)$, and the left-hand side

$$\frac{1}{2}\int_{\partial B_1(0)} \left|\mathbf{e}(x^1)\max(x\cdot\nu(x^1),0)^2 - \mathbf{e}(x^2)\max(x\cdot\nu(x^2),0)^2\right| d\mathcal{H}^{n-1}$$

$$\geq c(n)\,(|\nu(x^1) - \nu(x^2)| + |\mathbf{e}(x^1) - \mathbf{e}(x^2)|) \tag{30}$$

which can be seen as follows: Suppose first that for $j \to \infty$, $\nu_j^1 \to \bar\nu^1, \nu_j^2 \to \bar\nu^2, \mathbf{e}_j^1 \to \bar{\mathbf{e}}^1, \mathbf{e}_j^2 \to \bar{\mathbf{e}}^2$ such that

$$0 \leftarrow \int_{\partial B_1(0)} \left|\mathbf{e}_j^1\max(x\cdot\nu_j^1,0)^2 - \mathbf{e}_j^2\max(x\cdot\nu_j^2,0)^2\right| d\mathcal{H}^{n-1}.$$

We obtain

$$0 = \int_{\partial B_1(0)} \left|\bar{\mathbf{e}}^1\max(x\cdot\bar\nu^1,0)^2 - \bar{\mathbf{e}}^2\max(x\cdot\bar\nu^2,0)^2\right| d\mathcal{H}^{n-1},$$

implying immediately that $\bar{\mathbf{e}}^1 = \bar{\mathbf{e}}^2$ and that $\bar\nu^1 = \bar\nu^2$. Next, suppose towards a contradiction that, setting $c_j := |\nu_j^1 - \nu_j^2| + |\mathbf{e}_j^1 - \mathbf{e}_j^2|$, $(\nu_j^1 - \nu_j^2)/c_j \to \eta$, $(\mathbf{e}_j^1 - \mathbf{e}_j^2)/c_j \to \xi$ and

$$0 \leftarrow \frac{1}{c_j}\int_{\partial B_1(0)} \left|\mathbf{e}_j^1\max(x\cdot\nu_j^1,0)^2 - \mathbf{e}_j^2\max(x\cdot\nu_j^2,0)^2\right| d\mathcal{H}^{n-1} =: T_j.$$

We obtain

$$0 \leftarrow T_j \geq \frac{1}{c_j} \int_{\partial B_1(0)\cap\{x\cdot\nu_j^1>0\}\cap\{x\cdot\nu_j^2>0\}} \left|(\mathbf{e}_j^1 - \mathbf{e}_j^2)(x\cdot\nu_j^1)^2 \;+\; \mathbf{e}_j^2\, x\cdot(\nu_j^1+\nu_j^2)\, x\cdot(\nu_j^1-\nu_j^2)\right| \, d\mathcal{H}^{n-1}$$
$$\to \int_{\partial B_1(0)\cap\{x\cdot\bar\nu^1>0\}} \left|\xi(x\cdot\bar\nu^1)^2 \;+\; 2\bar{\mathbf{e}}^1\, x\cdot\bar\nu^1\, x\cdot\eta\right| \, d\mathcal{H}^{n-1}$$

as $j\to\infty$. We obtain $\xi = -2\bar\nu^1\cdot\eta\,\bar{\mathbf{e}}^1$, contradicting the fact that $0 = (|\mathbf{e}_j^1|^2 - |\mathbf{e}_j^2|^2)/c_j = ((\mathbf{e}_j^1+\mathbf{e}_j^2)\cdot(\mathbf{e}_j^1-\mathbf{e}_j^2)/c_j \to 2\bar{\mathbf{e}}^1\cdot\xi$ and thus proving (30).
Next, (29) as well as the regularity and non-degeneracy of $\mathbf{u}$ imply that for $\epsilon > 0$ there exists $\delta_2\in(0,\delta_1)$ such that for $x^1\in\mathcal{R}_{\mathbf{u}}\cap\overline{B_{\delta_1}(x^0)}$

$$\begin{aligned}&\mathbf{u}(y) = 0 \text{ for } y\in\overline{B_{\delta_2}(x^1)} \text{ satisfying } (y-x^1)\cdot\nu(x^1) < -\epsilon|y-x^1| \text{ and}\\ &|\mathbf{u}(y)|>0 \text{ for } y\in\overline{B_{\delta_2}(x^1)} \text{ satisfying } (y-x^1)\cdot\nu(x^1) > \epsilon|y-x^1| \;:\end{aligned}\tag{31}$$

assuming that (31) does not hold, we obtain a sequence $\mathcal{R}_{\mathbf{u}}\cap\overline{B_{\delta_1}(x^0)}\ni x^m \to \bar x$ and a sequence $y^m - x^m\to 0$ as $m\to\infty$ such that

$$\begin{aligned}&\text{either } |\mathbf{u}(y^m)|>0 \text{ and } (y^m-x^m)\cdot\nu(x^m) < -\epsilon|y^m-x^m|\\ &\text{or } \mathbf{u}(y^m) = 0 \text{ and } (y^m-x^m)\cdot\nu(x^m) > \epsilon|y^m-x^m|\,.\end{aligned}\tag{32}$$

On the other hand we know from (29) as well as from the regularity and non-degeneracy of the solution $\mathbf{u}$, that the sequence $\mathbf{u}_j(x) := \frac{\mathbf{u}(x^j+|y^j-x^j|\,x)}{|y^j-x^j|^2}$ converges in $C^{1,\alpha}_{\rm loc}(\mathbb{R}^n;\mathbb{R}^m)$ to $\frac{1}{2}\mathbf{e}(\bar x)\max(x\cdot\nu(\bar x),0)^2$ as $j\to\infty$ and that $\mathbf{u}_j = 0$ on each compact subset C of $\{x\cdot\nu(\bar x)<0\}$ provided that $j\geq j(C)$. This, however, contradicts (32) for large j.
Last, we use (31) in order to show that $\partial\{|\mathbf{u}|>0\}$ is for some $\delta_3\in(0,\delta_2)$ in $\overline{B_{\delta_3}(x^0)}$ the graph of a differentiable function: applying two rotations we may assume that $\nu(x^0) = \mathbf{e}^n$ and $\mathbf{e}(x^0) = \mathbf{e}^1$. Choosing now δ_2 with respect to $\epsilon = \frac{1}{2}$ and defining functions $g^+, g^- : B'_{\frac{\delta_2}{2}}(0)\to[-\infty,\infty]$, $g^+(x') := \sup\{x_n : x^0+(x',x_n)\in\partial\{|\mathbf{u}|>0\}\}$ and $g^-(x') := \inf\{x_n : x^0+(x',x_n)\in\partial\{|\mathbf{u}|>0\}\}$, we first note that (applying (31) at x^0), that $g^+ < +\infty$ and $g^- > -\infty$ on $B'_{\frac{\delta_2}{2}}(0)$. As $\nabla\mathbf{u}(x^0+(x',g^-(x'))) = 0$ for every $x'\in B'_{\frac{\delta_2}{2}}(0)$, we infer from Corollary 2 that $x^0+(x',g^-(x'))\in\mathcal{R}_{\mathbf{u}}$. It follows that (31) is applicable at $x^0+(x',g^-(x'))) = 0$ for every $x'\in B'_{\frac{\delta_2}{2}}(0)$, yielding by the Hölder continuity of $x^1\mapsto\nu(x^1)$ that for sufficiently small δ_3 the functions g^+ and g^- satisfy $g^+ = g^-$ Lipschitz-continuous on $\overline{B'_{\delta_3}(0)}$. In particular it follows that *all* free boundary points close to x^0 belong to $\mathcal{R}_{\mathbf{u}}$, and there are no other free boundary points (for example free boundary points with non-vanishing gradient) in a neighborhood of x^0.

Applying (31) once more with respect to arbitrary ϵ we see that g^+ is Fréchet-differentiable in $\overline{B'_{\delta_3}(0)}$, which finishes our proof in view of the already derived Hölder-continuity of the normal $\nu(x)$. □

Corollary 3 (Macroscopic criterion for regularity). *Let $\bar\alpha_n$ be the constant defined in Corollary 1. Then $B_{2r}(x^0)\subset D$, $x^0\in D\cap\partial\{|\mathbf{u}|>0\}$ and $W(\mathbf{u},x^0,r)<\bar\alpha_n$ imply that $\partial\{|\mathbf{u}|>0\}$ is in an open neighborhood of x^0 a $C^{1,\beta}$-surface.*

Proof. By $C^{1,\beta}$-regularity of $\mathbf{u}$ and Theorem 5, it suffices to show that for $W(\mathbf{u},x^0,r)<\bar\alpha_n$ either (i) $\nabla\mathbf{u}(x^0)\neq 0$ or (ii) $x^0\in\mathcal{R}_{\mathbf{u}}$. If both (i) and (ii) fail then by Lemma 1, Lemma 2 and Corollary 1, $W(\mathbf{u},x^0,r)\geq\lim_{r\to 0}W(\mathbf{u},x^0,r)\geq\bar\alpha_n$, contradicting the assumption. □

9. Appendix

Lemma 4. *Let Δ' be the Laplace-Beltrami operator on the unit sphere in $\mathbb{R}^n$, let the domain $\Omega' \subset \partial B_1(0) \subset \mathbb{R}^n$, let $\mathcal{L} := -\Delta' + q$ where $q \in C^0(\Omega')$ such that $q \geq q_0 > 0$ in Ω', and let $\lambda_k(\mathcal{L}, \Omega')$ denote the k-th eigenvalue with respect to the eigenvalue problem*

$$\mathcal{L}v = \lambda v \text{ in } \Omega'$$
$$v = 0 \text{ on } \partial\Omega';$$

here $\partial\Omega'$ denotes the boundary of Ω' relative to ∂B_1.
1. If $\tilde{\Omega}' \subset \Omega'$ then $\lambda_k(\mathcal{L}, \tilde{\Omega}') \geq \lambda_k(\mathcal{L}, \Omega')$ for every $k \in \mathbb{N}$. For $k = 1$ the inequality is strict.
2. $\lambda_k(\mathcal{L}, \Omega') \geq q_0 + \lambda_k(-\Delta', \Omega')$ for every $k \in \mathbb{N}$; in case $q \not\equiv q_0$ the inequality becomes a strict inequality.
3. $q = 1/h$ and $\Omega' \subset \partial B_1 \cap \{x_n > 0\}$ and $v \in W^{1,2}(\partial B_1)$ being an eigenfunction with respect to Ω' and $\lambda = 2n$ imply $v = ah$ for some real number $a \neq 0$. Here $h(x) = \frac{1}{2}\max(x_n, 0)^2$.
4. $\Omega' \subset \partial B_1(0) \cap \{x_n > -\delta(n, q_0)\}$ and $v \in W^{1,2}(\partial B_1(0))$ being an eigenfunction of $\mathcal{L}$ with respect to $\lambda = 2n$ and a domain Ω' imply $v = af_{\Omega'}$ for a real number $a \neq 0$ and a function $f_{\Omega'}$ which is positive on Ω' and depends only on Ω'.

Proof. 1. It suffices to remark that $v \in W_0^{1,2}(\tilde{\Omega}')$ implies $v \in W_0^{1,2}(\Omega')$, after extending v by zero outside $\tilde{\Omega}'$.

2. Let $\mathcal{M}_{k-1}$be a subspace of $W_0^{1,2}(\Omega')$ of codimension $k-1$,

$$\mu(\mathcal{L}, \Omega', \mathcal{M}_{k-1}) \; = \inf_{v \in \mathcal{M}_{k-1}, \|v\|_{L^2(\Omega')}=1} \int_{\Omega'} (|\nabla' v|^2 + qv^2).$$

Due to the Courant minimax principle we have

$$\lambda_k(\mathcal{L}, \Omega') \; = \; \sup \; \mu(\mathcal{L}, \Omega', \mathcal{M}_{k-1})$$

where sup is taken over the set of all possible $\mathcal{M}_{k-1}$. Since

$$\mu(\mathcal{L}, \Omega', \mathcal{M}_{k-1}) \geq \inf_{v \in \mathcal{M}_{k-1}, \|v\|_{L^2(\Omega')}=1} \int_{\Omega'} |\nabla' v|^2 + \inf_{v \in \mathcal{M}_{k-1}; \|v\|_{L^2(\Omega')}=1} \int_{\Omega'} qv^2 \geq q_0 + \mu(-\Delta', \Omega', \mathcal{M}_{k-1}),$$

we may take $\mathcal{M}_{k-1} := \left\{ v \in W_0^{1,2}(\Omega') \; : \; \int_\Omega' vw_i = 0, i \leq k-1 \right\}$, where w_i is an eigenfunction with respect to the ith eigenvalue of $-\Delta'$ on Ω'. For such $\mathcal{M}_{k-1}$ we obtain $\mu(-\Delta', \Omega', \mathcal{M}_{k-1}) = \lambda_k(-\Delta', \Omega')$, and 2. is proved.

3. In this case the eigenvalue problem on the sphere becomes

$$\begin{aligned} &-\Delta' v + \frac{2}{\cos^2\theta} v = 2nv \text{ in } \Omega' \subset \{x_n > 0\}, \\ &v = 0 \text{ on } \partial\Omega'. \end{aligned}$$

Since $2n = \lambda_2(-\Delta', \partial B_1(0) \cap \{x_n > 0\})$, we obtain from 1. and 2. that $\lambda_2(\mathcal{L}, \Omega') > 2 + 2n$. But then v must be an eigenfunction with respect to the *first* eigenvalue $\lambda_1(\mathcal{L}, \Omega')$, and $\lambda_1(\mathcal{L}, \Omega') = 2n$. Observe now that h is an eigenfunction with respect to the first eigenvalue $\lambda_1(\mathcal{L}, \partial B_1(0) \cap \{x_n > 0\})$, so that

$$\lambda_1(\mathcal{L}, \partial B_1(0) \cap \{x_n > 0\}) = 2n = \lambda_1(\mathcal{L}, \Omega').$$

This is only possible if $\Omega' = \partial B_1(0) \cap \{x_n > 0\}$ and $v = ah$ for some $a \neq 0$.

4. Observe that the second eigenvalue of $-\Delta'$ in a half-sphere is $2n$, therefore due to 1. and continuity of λ_2 with respect to the size of a spherical cap, $\lambda_2(-\Delta', \Omega') \geq 2n - \omega(n, \delta)$ where $\omega(n, \delta) \to 0$ as $\delta \to 0$. From 2. it follows that $\lambda_2(\mathcal{L}, \Omega') \geq 2n - \omega(n, \delta) + q_0 > 2n$ if $\delta = \delta(n, q_0)$ is small. Thus, v must be an eigenfunction with respect to the *first* eigenvalue

$\lambda_1(\mathcal{L}, \Omega')$, and $\lambda_1(\mathcal{L}, \Omega') = 2n$. Again, the eigenspace is a one-dimensional space such that $v = af_{\Omega'}$ for some real number $a \neq 0$ and a function $f_{\Omega'}$ depending only on Ω'; We also know that first eigenfunctions do not change sign in the connected set Ω'. □

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