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Abstract

In this note we are concerned with interior regularity properties of the p-Poisson problem $\Delta_p(u) = f$ with $p > 2$. For all $0 < \lambda \leq 1$ we constuct right-hand sides f of differentiability $-1 + \lambda$ such that the (Besov-) smoothness of corresponding solutions u is essentially limited to $1 + \lambda/(p-1)$. The statements are of local nature and cover all integrability parameters. They particularly imply the optimality of a shift theorem due to Savaré [J. Funct. Anal. 152:176–201, 1998], as well as of some recent Besov regularity results of Dahlke et al. [Nonlinear Anal. 130:298–329, 2016].

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1 Introduction and main results

In what follows we deal with interior regularity properties of solutions $u \in W_p^1(\Omega)$ to the p-Poisson problem

$$-\Delta_p(u) = f \tag{1}$$

on bounded Lipschitz domains $\Omega \subset \mathbb{R}^d$ for $d \in \mathbb{N}$ and $1 < p < \infty$. Here the p-Laplace operator Δ_p is given by

$$\Delta_p(u) := \operatorname{div}(A(\nabla u)), \qquad \text{where} \qquad A(\nabla u) := |\nabla u|^{p-2} \, \nabla u$$

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is called the naturally associated vector field. For distributions $f \in W^{-1}_{p'}(\Omega) := (W^1_{p,0}(\Omega))'$, with $1/p + 1/p' = 1$, the corresponding variational formulation is given by

$$\int_\Omega \langle A(\nabla u)(x), \nabla\psi(x)\rangle_{\mathbb{R}^d} \, \mathrm{d}x = f(\psi), \qquad \psi \in \mathcal{D}(\Omega),$$

where $\mathcal{D}(\Omega) := C_0^\infty(\Omega)$ is the set of test functions on Ω, the space $W^1_{p,0}(\Omega)$ is the closure of $\mathcal{D}(\Omega)$ w.r.t. the first order L_p-Sobolev norm, and $\langle\cdot,\cdot\rangle_{\mathbb{R}^d}$ denotes the usual inner product on $\mathbb{R}^d$.

Equations of type (1) arise in various applications such as non-Newtonian fluid theory, rheology, radiation of heat and many others. In fact the quasi-linear operator Δ_p has a similar model character for nonlinear problems as the ordinary Laplacian (i.e., the case $p = 2$) for linear problems. Meanwhile, many results concerning existence and uniqueness of solutions are known. For details we refer to [16] and the references therein. However, most of these results deal with fairly classical function spaces like Hölder spaces and BMO, or Lebesgue-Sobolev spaces of non-fractional smoothness $s \in \mathbb{N}_0$; see, e.g., [4, 5] and [6, 13]. On the other hand, in view of strong relations to nonlinear approximation classes and adaptive numerical algorithms, regularity results in more general Sobolev-Slobodeckij and Besov type spaces of fractional smoothness became more and more important in recent times; cf. [8, 11]. For the p-Poisson equation (1) and related quasi-linear problems only few results are known in this direction, see [3, 9, 12], as well as [1, 10, 19, 21].

Let us recall that for $0 < \varrho, q \leq \infty$ and $s \in \mathbb{R}$ Besov spaces $B^s_{\varrho,q}(\mathbb{R}^d)$ on the whole of $\mathbb{R}^d$ are quasi-Banach spaces which can be defined as subsets of tempered distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ by means of harmonic analysis. The corresponding spaces on domains Ω (i.e. connected open subsets of $\mathbb{R}^d$) are then defined by restriction such that we obtain subsets of $\mathcal{D}'(\Omega)$, the topological dual of $\mathcal{D}(\Omega)$.

Remark 1.1 (Function spaces)**.** We assume that the reader is familiar with the basics of function space theory as it can be found, e.g., in the monographic series of Triebel [23, 24, 25, 26] or in [7, Appendix A]. Anyhow, let us mention that by now various equivalent characterizations, embeddings, interpolation and duality assertions for the scale of Besov spaces are known. Without going into further details, let us recall the following results, valid for bounded Lipschitz domains $\Omega \subset \mathbb{R}^d$ or $\Omega = \mathbb{R}^d$ itself:

(i) For $0 < \varrho, q \leq \infty$, and $s > d\max\{0, 1/\varrho - 1\}$ the Besov space $B^s_{\varrho,q}(\Omega)$ can be characterized as collection of all $g \in L_\varrho(\Omega)$ for which

$$|g|_{B^s_{\varrho,q}(\Omega)} := \begin{cases} \left(\int_0^{\bar{t}} \left[t^{-s} \sup_{\substack{h\in\mathbb{R}^d,\\ |h|_2\leq t}} \left\| \Delta_h^k g \,\middle|\, L_\varrho(\Omega_{h,k}) \right\| \right]^q \frac{\mathrm{d}t}{t} \right)^{1/q}, & 0 < q < \infty, \\ \sup_{0<t\leq\bar{t}} t^{-s} \sup_{\substack{h\in\mathbb{R}^d,\\ |h|_2\leq t}} \left\| \Delta_h^k g \,\middle|\, L_\varrho(\Omega_{h,k}) \right\|, & q = \infty, \end{cases}$$

is finite [25, Sect. 1.11.9]. In fact, the expression

$$\left\| g \,\middle|\, B^s_{\varrho,q}(\Omega) \right\| := \| g \,|\, L_\varrho(\Omega) \| + |g|_{B^s_{\varrho,q}(\Omega)}$$

provides a quasi-norm on $B^s_{\varrho,q}(\Omega)$. Here we assume $\bar{t} > 0$ and $k > s$ to be fixed. Further, $\Delta_h^k g$ denotes the k-th order finite difference of g with step size $h \in \mathbb{R}^d$ and

$$\Omega_{h,k} := \{ x \in \mathbb{R}^d \,|\, x + \ell h \in \Omega \text{ for all } \ell = 0, \ldots, k \}.$$

(ii) For $0 < s \notin \mathbb{N}$ we have $B^s_{\infty,\infty}(\Omega) = C^s(\Omega)$ (Hölder spaces) and

$$W^s_\varrho(\Omega) = B^s_{\varrho,\varrho}(\Omega) \quad \text{for all} \quad 1 \leq \varrho < \infty,$$

(Sobolev-Slobodeckij spaces) in the sense of equivalent norms.

So, roughly speaking, in $B^s_{\varrho,q}(\Omega)$ we collect all g such that their weak partial derivatives $D^\alpha g$ up to order s belong to the Lebesgue space $L_\varrho(\Omega)$. The third parameter $0 < q \leq \infty$ acts as a minor important fine index. □

In his seminal paper [19] Savaré developed a variational argument which allows to show the following shift theorem for the p-Poisson problem:

Proposition 1.2 (see [19, Thm. 2])**.** *For $d \in \mathbb{N}$ let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. For $2 < p < \infty$ and $f \in W^{-1}_{p'}(\Omega)$ let u be the unique weak solution to* (1) *in $W^1_{p,0}(\Omega)$. Then for all $\lambda \in (0, 1/p')$ the following implications hold:*

$$f \in W^{-1+\lambda}_{p'}(\Omega) \quad \Longrightarrow \quad u \in W_p^{1+\frac{\lambda}{p-1}}(\Omega) \tag{2}$$

and

$$f \in B^{-1+\frac{1}{p'}}_{p',1}(\Omega) \quad \Longrightarrow \quad u \in B^{1+\frac{1/p'}{p-1}}_{p,\infty}(\Omega).$$

In addition, Savaré argues that (2) is "optimal" [19, Rem. 4.3] and refers to Simon [21]. But Simon's optimality results refer to a (slightly) different equation on the whole of $\mathbb{R}^d$ with right hand sides in L_r or C^∞ and hence they unfortunately do not cover Savaré's claim. However, as we will see in Theorem 1.3 below, it is possible to use similar ideas in order to show that for $p > 2$ and all $0 < \lambda < 1$ there are right-hand sides of smoothness $-1 + \lambda$ such that the smoothness of corresponding solutions to the p-Poisson problem (1) is essentially limited by $1 + \lambda/(p-1)$. Moreover, this actually holds *independently* of the integrability parameter. That is, in sharp contrast to point singularities, we *do not gain smoothness* when derivatives are measured in weaker L_ϱ-norms!

For a (constructive) proof of the following main result of this paper we refer to Section 2.2 below.

Theorem 1.3. *Let $d \in \mathbb{N}$ and $2 \leq p < \infty$ be fixed. Further let $\Omega \subseteq \mathbb{R}^d$ be either $\mathbb{R}^d$ itself, a bounded Lipschitz domain, or an interval (if $d = 1$). Moreover, assume $0 < \varepsilon < 1/p$. Then for all $\varepsilon\,(p-1) < \lambda < 1 - \varepsilon$ and $1 < \mu \leq \infty$ there exists a right-hand side*

$$f = f_{\lambda,\mu} \in B^{-1+\lambda}_{\mu,\infty}(\Omega) \cap W^{-1}_{p'}(\Omega)$$

with compact support in Ω such that the corresponding weak solution $u \in W^1_{p,0}(\Omega)$ to (1) is compactly supported as well and satisfies

$$u \in \begin{cases} B^{1+\frac{\lambda}{p-1}-\varepsilon}_{\varrho,q}(\Omega) \setminus B^{1+\frac{\lambda}{p-1}}_{\varrho,q}(\Omega) & \text{if } \mu(p-1) < \varrho \leq \infty \text{ and } 0 < q \leq \infty, \\ B^{1+\frac{\lambda}{p-1}}_{\varrho,\infty}(\Omega) \setminus B^{1+\frac{\lambda}{p-1}}_{\varrho,q}(\Omega) & \text{if } \varrho = \mu(p-1) \text{ and } 0 < q < \infty, \\ B^{1+\frac{\lambda}{p-1}}_{\varrho,q}(\Omega) \setminus B^{1+\frac{\lambda}{p-1}+\varepsilon}_{\varrho,q}(\Omega) & \text{if } 1 \leq \varrho < \mu(p-1) \text{ and } 0 < q \leq \infty. \end{cases} \tag{3}$$

In addition, the naturally associated vector field satisfies

$$A(\nabla u) \in \begin{cases} \left(B^{\lambda-\varepsilon}_{\varrho,q}(\Omega) \setminus B^{\lambda}_{\varrho,q}(\Omega)\right)^d & \text{if } \mu < \varrho \leq \infty \text{ and } 0 < q \leq \infty, \\ \left(B^{\lambda}_{\varrho,\infty}(\Omega) \setminus B^{\lambda}_{\varrho,q}(\Omega)\right)^d & \text{if } \varrho = \mu \text{ and } 0 < q < \infty, \\ \left(B^{\lambda}_{\varrho,q}(\Omega) \setminus B^{\lambda+\varepsilon}_{\varrho,q}(\Omega)\right)^d & \text{if } 1 \leq \varrho < \mu \text{ and } 0 < q \leq \infty. \end{cases} \tag{4}$$

Before we proceed some general comments are in order:

Remark 1.4. First of all, let us stress the point that, due to the compact support of f and u, Theorem 1.3 is of *local* nature.

Moreover, we note that the restriction to $\varrho \geq 1$ is for notational convenience only. Using standard embeddings (see Proposition 2.2iii below) and complex interpolation (see, e.g., Kalton *et al.* [15, Theorem 5.2]) we can easily extend (3) to

$$u \in B^{1+\frac{\lambda}{p-1}}_{\varrho,q}(\Omega) \setminus B^{1+\frac{\lambda}{p-1}+c_\varrho\varepsilon}_{\varrho,q}(\Omega) \qquad \text{for all} \qquad 0 < \varrho < 1 \qquad \text{and} \qquad 0 < q \leq \infty$$

with some $c_\varrho \sim 1/\varrho$. Likewise, the same arguments can be used to extend also (4). □

Observe that Theorem 1.3 applied for $\mu := p'$ indeed shows optimality of Savaré's result in some sense:

Corollary 1.5. *In Proposition 1.2 the smoothness of u w.r.t. L_p cannot be improved without strengthening the assumptions on f.*

Proof. Choosing $\lambda := \widetilde{\lambda} + (p-1)\delta$ with some $\widetilde{\lambda} \in (0, 1/p')$ and $\delta > 0$ arbitrarily small, Theorem 1.3 allows to find a right-hand side $f \in B^{-1+\lambda}_{\mu,\infty}(\Omega) \hookrightarrow B^{-1+\widetilde{\lambda}}_{p',p'}(\Omega) = W^{-1+\widetilde{\lambda}}_{p'}(\Omega)$ such that the corresponding solution of the Dirichlet problem for the p-Poisson equation satisfies $u \notin B^{1+\frac{\widetilde{\lambda}}{p-1}+\delta}_{p,p}(\Omega) = W^{1+\frac{\widetilde{\lambda}}{p-1}+\delta}_{p}(\Omega)$. Similarly for $\lambda := 1/p' + \delta$ with $\delta > 0$, there exists $f \in B^{-1+\lambda}_{\mu,\infty}(\Omega) \hookrightarrow B^{-1+\frac{1}{p'}}_{p',1}(\Omega)$ such that $u \notin B^{1+\frac{\lambda}{p-1}}_{p,q}(\Omega) \hookleftarrow B^{1+\frac{1/p'}{p-1}+\delta}_{p,\infty}(\Omega)$. In view of Remark 1.4, these examples remain valid also on smooth domains. ■

Furthermore, Theorem 1.3 shows that regarding regularity questions it seems better to look at the the mapping $f \mapsto A(\nabla u)$, rather than $f \mapsto u$. In fact, in view of the case $\varrho = \mu$ in (4), one might conjecture the existence of a p-independent mechanism which (for some range of parameters) locally transfers exactly one order of regularity from the right-hand side f to the naturally associated vector field $A(\nabla u)$. For the case $d = 2$ this already has been verified in [2, 3]. In other words, Theorem 1.3 shows that also their results cannot be improved. Let us note in passing that in the linear case ($p = 2$) the vector field $A(\nabla u)$ reduces to ∇u. In this situation, it is well-known that $f \in W^{s-1}_{\varrho,\mathrm{loc}}(\Omega)$ implies $u \in W^{s+1}_{\varrho,\mathrm{loc}}(\Omega)$ and hence $\nabla u \in \left(W^{s}_{\varrho,\mathrm{loc}}(\Omega)\right)^d$, at least for some $1 < \varrho < \infty$ and $s \geq 0$. However, the seminal work of Jerison and Kenig [14] shows that there are C^1 domains for which a global analogue of this shift mechanism fails if the smoothness parameter s is too large w.r.t. the integrability ϱ. For corresponding assertions in Besov spaces, see, e.g., [17, Thm. 3.1]. Further, we refer to [7, Sect. 3.1] for an extensive discussion of these results.

Theorem 1.3 is complemented by

Theorem 1.6. *Let $d \in \mathbb{N}$ and $2 < p < \infty$. Further let $\Omega \subseteq \mathbb{R}^d$ be either $\mathbb{R}^d$ itself, a bounded Lipschitz domain, or an interval (if $d = 1$). Moreover, let $0 < \varepsilon < \min\{1/(p-1), 1-1/(p-1)\}$. Then for all $1 < \mu \leq \infty$ there exists a compactly supported right-hand side*

$$f = f_\mu \in L_\nu(\Omega) \cap W^{-1}_{p'}(\Omega) \quad \textit{for all} \quad \begin{cases} 0 < \nu < \mu & \textit{if} \quad \mu < \infty, \\ 0 < \nu \leq \infty & \textit{if} \quad \mu = \infty, \end{cases}$$

such that the corresponding weak solution $u \in W^1_{p,0}(\Omega)$ to (1) is compactly supported as well and satisfies (3) with $\lambda = 1$. Moreover, then for $1 < \varrho < \infty$ there holds

$$A(\nabla u) \in \left(W^1_\varrho(\Omega)\right)^d \qquad \textit{if and only if} \qquad \varrho < \mu.$$

Here Remark 1.4 applies likewise. Moreover, also this result implies certain optimality statements:

Remark 1.7. At first, let us mention [6, Thm. 2.4] which shows that under some boundary regularity assumptions $f \in L_2(\Omega)$ implies $A(\nabla u) \in \left(W_2^1(\Omega)\right)^d$.

Secondly, setting $\mu := \varrho := \infty$ in Theorem 1.6, we recover the well-known assertion that for bounded right-hand sides the local Hölder regularity of the gradient ∇u of solutions to the p-Poisson equation (1) with $p > 2$ is bounded by $1/(p-1)$.

Last but not least, in [9] it has been shown that for $p > 2$, bounded Lipschitz domains $\Omega \subset \mathbb{R}^2$, and right-hand sides $f \in L_\infty(\Omega)$ the unique solution $u \in W_{p,0}^1(\Omega)$ to (1) satisfies

$$u \in B^\sigma_{\tau_\sigma,\tau_\sigma}(\Omega) \qquad \text{for all} \qquad 0 < \sigma < \overline{\sigma} := 1 + \frac{1}{p-1} \qquad \text{and} \qquad \frac{1}{\tau_\sigma} = \frac{\sigma}{2} + \frac{1}{p}.$$

In view of Theorem 1.6 and Remark 1.4, $\overline{\sigma}$ cannot be replaced by any larger number. □

The rest of this note is devoted to the proofs of Theorems 1.3 and 1.6, respectively. Section 2.1 collects some quite technical preparations. Afterwards, the statements are proven easily in Sections 2.2 and 2.3.

Notations: In the sequel $\mathbb{N}$ denotes the natural numbers without zero and we use $\mathbb{R}_+$ for the set of strictly positive real numbers. For families $\{a_j \mid j \in \mathcal{J}\}$ and $\{b_j \mid j \in \mathcal{J}\}$ of non-negative real numbers over a common index set $\mathcal{J}$ we write $a_j \lesssim b_j$ if there exists a constant $c > 0$ (independent of the context-dependent parameters j) such that $a_j \leq c \cdot b_j$ holds uniformly in $j \in \mathcal{J}$. Consequently, $a_j \sim b_j$ means $a_j \lesssim b_j$ and $b_j \lesssim a_j$. In addition, the symbol $\hookrightarrow$ is used to denote continuous embeddings.

2 Proofs

Our main proofs given in Sections 2.2 and 2.3 below require some preparations. The basic idea will be based on a construction given by Simon [22]; cf. also Remark 2.4 below.

2.1 Preparations

For $1 < \theta < \infty$ define the sequence

$$a_{n,\theta} := 4 \sum_{j=1}^{n-1} j^{-\theta}, \qquad n \in \mathbb{N} \setminus \{1\}.$$

Then for all $n \geq 3$

$$4 = a_{2,\theta} < \ldots < a_{n,\theta} < a_{n+1,\theta} = a_{n,\theta} + 4\, n^{-\theta} < \ldots < a_{\infty,\theta} := \lim_{n\to\infty} a_{n,\theta} = 4\,\zeta(\theta) < \infty,$$

where ζ denotes the Riemann zeta function. Further, with $\sigma \in \mathbb{R}_+$ let $w_{\sigma,\theta}\colon \mathbb{R} \to [0,\infty)$ be defined piecewise by

$$w_{\sigma,\theta}(\xi) := \begin{cases} (\xi - a_{n,\theta})^\sigma & \text{if} \quad a_{n,\theta} \leq \xi < a_{n,\theta} + n^{-\theta}, \\ n^{-\theta\sigma} & \text{if} \quad a_{n,\theta} + n^{-\theta} \leq \xi < a_{n,\theta} + 2\, n^{-\theta}, \\ (a_{n,\theta} + 3\, n^{-\theta} - \xi)^\sigma & \text{if} \quad a_{n,\theta} + 2\, n^{-\theta} \leq \xi < a_{n,\theta} + 3\, n^{-\theta}, \\ 0 & \text{if} \quad a_{n,\theta} + 3\, n^{-\theta} \leq \xi < a_{n+1,\theta}, \end{cases} \tag{5}$$

on $[a_{n,\theta}, a_{n+1,\theta})$, $n \geq 2$, and

$$w_{\sigma,\theta}(\xi) := 0 \quad \text{on} \quad \mathbb{R} \setminus [a_{2,\theta}, a_{\infty,\theta}).$$

Moreover, let us define $\mathscr{S}_\theta := [4, 4\,\zeta(\theta)] \subset \mathbb{R}$, as well as the set of transition points

$$\mathscr{P}_\theta := \left\{ \xi \in [a_{2,\theta}, a_{\infty,\theta}) \,\middle|\, \xi = a_{n,\theta} + k\, n^{-\theta} \text{ for some } n \geq 2 \text{ and } k \in \{0, \ldots, 3\} \right\} \cup \{a_{\infty,\theta}\} \subset \mathbb{R}.$$

Lemma 2.1 (Properties of $w_{\sigma,\theta}$)**.** *Let $\sigma \in \mathbb{R}_+$ and $1 < \theta < \infty$, as well as $0 < \varrho \leq \infty$. Then*

(i) $w_{\sigma,\theta}$ is continuous with compact support $\operatorname{supp}(w_{\sigma,\theta}) \subseteq \mathscr{S}_\theta$*.*

(ii) $\mathscr{P}_\theta$ is countable and $w_{\sigma,\theta}$ is continuously differentiable on $\mathbb{R} \setminus \mathscr{P}_\theta$, i.e., $w'_{\sigma,\theta}$ exists almost everywhere.

(iii) $w_{\gamma\sigma,\theta} = (w_{\sigma,\theta})^{\gamma-1} w_{\sigma,\theta}$ for all $\gamma \in \mathbb{R}_+$.

(iv) $0 \leq w_{\sigma,\theta}(\xi) \leq 2^{-\sigma\theta}$ for all $\xi \in \mathbb{R}$.

Proof. All statements are obvious consequences of the definition of $w_{\sigma,\theta}$. In order to see (iv), note that $0 \leq w_{\sigma,\theta}(\xi) \leq n^{-\sigma\theta}$ on $[a_{n,\theta}, a_{n+1,\theta}]$ with $n \geq 2$. ∎

In the sequel, we will need sharp regularity assertions for $w_{\sigma,\theta}$. Before we state and prove them, let us recall some more or less well-known embedding results for Besov spaces which are proven here for the sake of completeness.

Proposition 2.2 (Embeddings)**.** *For $d \in \mathbb{N}$ let $\Omega \subseteq \mathbb{R}^d$ be either $\mathbb{R}^d$ itself, a bounded Lipschitz domain, or an interval (if $d = 1$). Further let $0 < \varrho, \varrho_0, \varrho_1, q, q_0, q_1 \leq \infty$ and $s, s_0, s_1 \in \mathbb{R}$. Then* *(i) There holds* $B^{1+s}_{\varrho,q}(\Omega) = \left\{ g \in B^s_{\varrho,q}(\Omega) \;\middle|\; \nabla g \in (B^s_{\varrho,q}(\Omega))^d \right\}$ *with*

$$\left\| g \,\middle|\, B^{1+s}_{\varrho,q}(\Omega) \right\| \sim \sum_{\substack{\alpha \in \mathbb{N}_0^d,\\ |\alpha|_1 \leq 1}} \left\| D^\alpha g \,\middle|\, B^s_{\varrho,q}(\Omega) \right\|.$$

(ii) If $s_1 < s_0$*, then* $g \in B^{s_0}_{\varrho,q_0}(\Omega)$ *implies* $g \in B^{s_1}_{\varrho,q_1}(\Omega)$.

(iii) If $\varrho_1 \leq \varrho_0$ *and* $g \in B^s_{\varrho_0,q}(\Omega)$ *has compact support in* $\mathbb{R}^d$*, then* $g \in B^s_{\varrho_1,q}(\Omega)$.

Proof. Assertion (i) is a special instance of [26, Prop. 4.21].

So, let us prove (ii). By means of Rychkov's extension operator [18] we can w.l.o.g. assume that $\Omega = \mathbb{R}^d$. Further, we let $c(g)$ denote the sequence of wavelet coefficients of g w.r.t. a sufficiently smooth Daubechies wavelet system on $\mathbb{R}^d$. Then the wavelet isomorphism from [25, Thm. 3.5] implies that $\left\| g \,\middle|\, B^s_{\varrho,q}(\mathbb{R}^d) \right\| \sim \left\| c(g) \,\middle|\, b^s_{\varrho,q}(\nabla) \right\|$ for all $0 < \varrho, q \leq \infty$ and $s \in \mathbb{R}$ with $b^s_{\varrho,q}(\nabla)$ being suitable sequence spaces. Now (ii) follows from the standard embedding $b^{s_0}_{\varrho_0,q_0}(\nabla) \hookrightarrow b^{s_1}_{\varrho_0,q_1}(\nabla)$ if $s_1 < s_0$ which can be found, e.g., in [27, Prop. 2.5].

In order to prove assertion (iii), we note that the compact support of g implies that $\left\| c(g) \,\middle|\, b^s_{p,q}(\nabla) \right\|$ equals $\left\| c(g)\big|_{\widetilde{\nabla}} \,\middle|\, b^s_{p,q}(\widetilde{\nabla}) \right\|$, where $b^s_{p,q}(\widetilde{\nabla})$ refers to the corresponding sequence space for some bounded domain. For these spaces there holds the embedding $b^{s_0}_{\varrho_0,q_0}(\widetilde{\nabla}) \hookrightarrow b^{s_0}_{\varrho_1,q_0}(\widetilde{\nabla})$ if $\varrho_1 \leq \varrho_0$, see [27, Prop. 2.5] again. ■

Lemma 2.3 (Regularity of $w_{\sigma,\theta}$)**.** *Let* $\sigma \in \mathbb{R}_+$ *and* $1 < \theta < \infty$*, as well as* $0 < \varrho \leq \infty$*. Then*

(i) $w_{\sigma,\theta} \in L_\varrho(\mathbb{R})$,

(ii) $w'_{\sigma,\theta} \in L_\varrho(\mathbb{R})$ *holds if and only if*

$$\sigma \geq 1, \qquad \text{or} \qquad 0 < \sigma < 1 \quad \text{and} \quad \frac{1-\sigma}{1-1/\theta} < \frac{1}{\varrho}. \tag{6}$$

(iii) Additionally assume $0 < \sigma < 1/\theta < 1$ *and*

$$0 \leq \frac{1}{\varrho} < \min\left\{ \theta\,(1+\sigma), \frac{1-\sigma}{1-1/\theta} \right\}. \tag{7}$$

Then $w_{\sigma,\theta} \in B^s_{\varrho,q}(\mathbb{R})$ *holds if and only if*

$$s = \sigma + \frac{1-1/\theta}{\varrho} \quad \text{and} \quad q = \infty, \qquad \text{or} \qquad s < \sigma + \frac{1-1/\theta}{\varrho} \quad \text{and} \quad 0 < q \leq \infty.$$

Proof. In view of Lemma 2.1(i) assertion (i) is obvious.

Let us show (ii). Clearly $w'_{\sigma,\theta} \in L_\infty(\mathbb{R})$ is equivalent to $\sigma \geq 1$. On the other hand, for $0 < \varrho < \infty$ we have

$$\big\| w'_{\sigma,\theta} \,\big|\, L_\varrho(\mathbb{R}) \big\|^\varrho = \sum_{n=2}^\infty \left(\int_{a_{n,\theta}}^{a_{n,\theta}+n^{-\theta}} \big| w'_{\sigma,\theta}(\xi) \big|^\varrho \, \mathrm{d}\xi + \int_{a_{n,\theta}+2\,n^{-\theta}}^{a_{n,\theta}+3\,n^{-\theta}} \big| w'_{\sigma,\theta}(\xi) \big|^\varrho \, \mathrm{d}\xi \right)$$
$$= 2\,\sigma^\varrho \sum_{n=2}^\infty \int_0^{n^{-\theta}} x^{(\sigma-1)\varrho} \, \mathrm{d}x.$$

The latter integral is finite if and only if $1 + (\sigma - 1)\varrho > 0$. In this case, there holds

$$\big\| w'_{\sigma,\theta} \,\big|\, L_\varrho(\mathbb{R}) \big\|^\varrho \sim \sum_{n=2}^\infty n^{-\theta(1+(\sigma-1)\varrho)} = \zeta\big(\theta(1 + (\sigma - 1)\varrho)\big) - 1$$

which is finite if only if the argument of the zeta function ζ is strictly larger than one. Thus, for $0 < \varrho < \infty$ we have $w'_{\sigma,\theta} \in L_\varrho(\mathbb{R})$ if and only if $1 + (\sigma - 1)\varrho > 0$ and $\theta(1 + (\sigma - 1)\varrho) > 1$ which is equivalent to

$$\max\left\{ 1 - \sigma, \frac{1-\sigma}{1 - 1/\theta} \right\} < \frac{1}{\varrho}. \tag{8}$$

For $\sigma \geq 1$ this condition holds for all ϱ. On the other hand, if $0 < \sigma < 1$, then $0 < 1 - \sigma < 1$ and $1/(1 - 1/\theta) = \theta/(\theta - 1) > 1$ implies that the maximum in (8) is attained by its second entry. Hence, $w'_{\sigma,\theta} \in L_\varrho(\mathbb{R})$ is equivalent to (6).

It remains to show assertion (iii). We split its proof into several steps.

Step 1 (Preparations). Note that for (iii) it suffices to show that $w_{\sigma,\theta} \in B^s_{\varrho,\infty}(\mathbb{R}) \setminus B^s_{\varrho,q}(\mathbb{R})$ for

$$s = \sigma + \frac{1 - 1/\theta}{\varrho} \qquad \text{and all} \qquad q \in \mathbb{R}_+.$$

Indeed, according to Proposition 2.2(ii), $w_{\sigma,\theta} \in B^s_{\varrho,\infty}(\mathbb{R})$ implies $w_{\sigma,\theta} \in B^{s'}_{\varrho,q}(\mathbb{R})$ for all $s' < s$ and $0 < q \leq \infty$. Similarly, $w_{\sigma,\theta} \in B^{s''}_{\varrho,q}(\mathbb{R})$ for some $s'' > s$ and some $0 < q \leq \infty$ would yield $w_{\sigma,\theta} \in B^s_{\varrho,1}(\mathbb{R})$.

From (i) we know that $\| w_{\sigma,\theta} \,|\, L_\varrho(\mathbb{R}) \| < \infty$, so that it remains to prove that

$$|w_{\sigma,\theta}|_{B^s_{\varrho,q}(\mathbb{R})} < \infty \quad \text{if and only if} \quad q = \infty.$$

To this end, note that $0 < \sigma$ and $1/\theta < 1$ implies $s > 0$, while $1/\varrho < \theta\,(1 + \sigma)$ holds if and only if $s > 1/\varrho - 1$. Moreover, the assumption $1/\varrho < (1 - \sigma)/(1 - 1/\theta)$ is equivalent to $s < 1$. Hence,

$$\max\left\{ 0, \frac{1}{\varrho} - 1 \right\} < s < 1,$$

so that we can use first order differences. Therefore it is enough to show that

$$\|\Delta_h w_{\sigma,\theta}\,|\,L_\varrho(\mathbb{R})\| \sim |h|^{\sigma+(1-1/\theta)/\varrho} \qquad \text{for all} \quad h \in \mathbb{R} \quad \text{with} \quad |h| \leq (1/6)^\theta =: \bar{t}, \tag{9}$$

because then

$$t^{-s} \sup_{\substack{h\in\mathbb{R},\\ |h|\leq t}} \|\Delta_h w_{\sigma,\theta}\,|\,L_\varrho(\mathbb{R})\| \sim 1, \qquad 0 < t \leq \bar{t}.$$

Of course, we may assume w.l.o.g. that $h > 0$.

Step 2 (Case $\varrho = \infty$). We prove "$\lesssim$" for $\varrho = \infty$ in (9). For this purpose, it suffices to show that

$$|w_{\sigma,\theta}(x) - w_{\sigma,\theta}(y)| \leq 2\ |x-y|^\sigma \qquad \text{for all} \qquad x, y \in \mathbb{R} \quad \text{with} \quad x < y. \tag{10}$$

So let $x, y \in \mathbb{R}$ with $h := y-x > 0$ be fixed. Note that it is enough to consider $a_{2,\theta} \leq x < a_{\infty,\theta}$, because $x < a_{2,\theta}$ implies

$$|w_{\sigma,\theta}(x) - w_{\sigma,\theta}(y)| = |w_{\sigma,\theta}(x+h)| \leq h^\sigma \quad \text{for all} \quad h > 0,$$

while $x \geq a_{\infty,\theta}$ would lead to $w_{\sigma,\theta}(x) = w_{\sigma,\theta}(y) = 0$. For $x \in [a_{2,\theta}, a_{\infty,\theta})$ the quantity

$$M := M(x,\theta) := \max\{n \geq 2\,|\,a_{n,\theta} \leq x\}$$

is well-defined. In case $h = y - x \geq M^{-\theta}$, we have

$$|w_{\sigma,\theta}(x) - w_{\sigma,\theta}(y)| \leq |w_{\sigma,\theta}(x)| + |w_{\sigma,\theta}(y)| \leq 2\,(M^{-\theta})^\sigma \leq 2\,h^\sigma,$$

as claimed. So let us turn to the case $0 < h < M^{-\theta}$. If $y > a_{M+1,\theta}$, then again $w_{\sigma,\theta}(x) = 0$. Moreover, in this case $a_{M+1,\theta} < y = x + h < a_{M+1,\theta} + h$, i.e.,

$$|w_{\sigma,\theta}(x) - w_{\sigma,\theta}(y)| = |w_{\sigma,\theta}(y)| < h^\sigma.$$

Similarly, if $y \in [a_{M+1,\theta} - M^{-\theta}, a_{M+1,\theta}]$, then $w_{\sigma,\theta}(y) = 0$ and

$$|w_{\sigma,\theta}(x) - w_{\sigma,\theta}(y)| = |w_{\sigma,\theta}(x)| \leq h^\sigma.$$

Hence, we are left with the case $a_{M,\theta} \leq x < y < a_{M,\theta} + 3\,M^{-\theta}$ and $0 < h = y - x < M^{-\theta}$, but for this situation (10) is obvious.

For the corresponding lower bound let $0 < h < \bar{t}$. Then

$$\begin{aligned}
\|\Delta_h w_{\sigma,\theta}\,|\,L_\infty(\mathbb{R})\| &\geq \operatorname*{esssup}_{x\in(a_{2,\theta}-h,a_{2,\theta})} |w_{\sigma,\theta}(x+h) - w_{\sigma,\theta}(x)| \\
&= \operatorname*{esssup}_{y\in(a_{2,\theta},a_{2,\theta}+h)} |w_{\sigma,\theta}(y)| \\
&= w_{\sigma,\theta}(a_{2,\theta}+h) \\
&= h^\sigma.
\end{aligned}$$

Step 3 (Case $\varrho < \infty$). In order to prove (9) for $\varrho < \infty$, consider the disjoint union

$$\mathbb{R} = \mathscr{L}_\theta \cup \left(\bigcup_{n=2}^{\infty} (\mathscr{L}_{n,\theta} \cup \mathscr{T}_{n,\theta} \cup \mathscr{R}_{n,\theta}) \right) \cup \mathscr{R}_\theta,$$

where we set $\mathscr{L}_\theta := (-\infty, 4)$, $\mathscr{R}_\theta := [4\,\zeta(\theta), \infty)$, as well as

$$\mathscr{L}_{n,\theta} := [a_{n,\theta}, a_{n,\theta} + n^{-\theta}), \quad \mathscr{T}_{n,\theta} := [a_{n,\theta} + n^{-\theta}, a_{n,\theta} + 3\,n^{-\theta}),$$

and $\mathscr{R}_{n,\theta} := [a_{n,\theta} + 3\,n^{-\theta}, a_{n,\theta} + 4\,n^{-\theta})$ for all $n \in \mathbb{N}$ with $n \geq 2$. Now let $0 < h \leq \overline{t} = (1/6)^\theta$ be arbitrarily fixed. Then $N(h,\theta) := \lceil h^{-1/\theta}/3 \rceil \in \mathbb{N}$ satisfies $N(h,\theta) \geq 2$ and

$$N(h,\theta) - 1 \geq \frac{1}{3}\, h^{-1/\theta} - 1 = h^{-1/\theta} \left(\frac{1}{3} - h^{1/\theta} \right) \geq \frac{1}{6}\, h^{-1/\theta}$$

due to the assumption $\theta > 1$. Further, for all $n \in \mathbb{N}$ with $2 \leq n \leq N(h,\theta)$ it holds $n \leq h^{-1/\theta}/3 + 1$, i.e.,

$$0 < h \leq \frac{1}{3^\theta (n-1)^\theta} \leq (n+1)^{-\theta}.$$

In this case,

$$\begin{aligned}
\|\Delta_h w_{\sigma,\theta} \,|\, L_\varrho(\mathscr{R}_{n,\theta})\|^\varrho &= \int_{a_{n,\theta}+3\,n^{-\theta}}^{a_{n,\theta}+4\,n^{-\theta}} |w_{\sigma,\theta}(\xi + h) - w_{\sigma,\theta}(\xi)|^\varrho \,\mathrm{d}\xi \\
&= \int_{a_{n+1,\theta}}^{a_{n+1,\theta}+h} w_{\sigma,\theta}(y)^\varrho \,\mathrm{d}y \\
&= \int_0^h y^{\sigma\varrho} \,\mathrm{d}y \\
&= \frac{1}{\sigma\varrho + 1}\, h^{\sigma\varrho+1}
\end{aligned} \tag{11}$$

which yields the desired lower bound

$$\begin{aligned}
\|\Delta_h w_{\sigma,\theta} \,|\, L_\varrho(\mathbb{R})\|^\varrho &\geq \left\| \Delta_h w_{\sigma,\theta} \,\middle|\, L_\varrho \left(\bigcup_{n=2}^{N(h,\theta)} \mathscr{R}_{n,\theta} \right) \right\|^\varrho \\
&= \sum_{n=2}^{N(h,\theta)} \|\Delta_h w_{\sigma,\theta} \,|\, L_\varrho\,(\mathscr{R}_{n,\theta})\|^\varrho \\
&\geq (N(h,\theta) - 1)\, \frac{1}{\sigma\varrho + 1}\, h^{\sigma\varrho+1} \\
&\gtrsim h^{\sigma\varrho+1-1/\theta}.
\end{aligned}$$

Let us show the corresponding upper bound. Using Step 2 and log-convexity of L_ϱ-norms, we see that the bound for L_ϱ implies the respective bound for all L_p with $0 < 1/p < 1/\varrho$:

$$\begin{aligned}\|\Delta_h w_{\sigma,\theta} \,|\, L_p(\mathbb{R})\|^p &\leq \left(\|\Delta_h w_{\sigma,\theta} \,|\, L_\varrho(\mathbb{R})\|^{\varrho/p} \; \|\Delta_h w_{\sigma,\theta} \,|\, L_\infty(\mathbb{R})\|^{1-\varrho/p}\right)^p \\ &= \|\Delta_h w_{\sigma,\theta} \,|\, L_\varrho(\mathbb{R})\|^{\varrho} \; \|\Delta_h w_{\sigma,\theta} \,|\, L_\infty(\mathbb{R})\|^{p-\varrho} \\ &\lesssim h^{\varrho\sigma+1-1/\theta} \; (h^\sigma)^{p-\varrho} \\ &= h^{p\sigma+1-1/\theta}.\end{aligned}$$

Therefore, since $(1-\sigma)/(1-1/\theta) > 1$ if and only if $\sigma < 1/\theta$, we may assume w.l.o.g.

$$1 \leq \frac{1}{\varrho} < \frac{1-\sigma}{1-1/\theta}.$$

So, let $0 < \varrho \leq 1$ and consider $2 \leq n \leq N(h,\theta)$. Then Hölder's inequality (with $1/r := 1-\varrho$ and $1/r' = \varrho$) and the monotonicity of $w_{\sigma,\theta}$ imply

$$\begin{aligned}\|\Delta_h w_{\sigma,\theta} \,|\, L_\varrho(\mathscr{L}_{n,\theta})\| &\leq \left(n^{-\theta}\right)^{1/\varrho-1} \int_{a_{n,\theta}}^{a_{n,\theta}+n^{-\theta}} \big(w_{\sigma,\theta}(\xi+h) - w_{\sigma,\theta}(\xi)\big)\,\mathrm{d}\xi \\ &= n^{-\theta(1/\varrho-1)} \left(\int_{a_{n,\theta}+h}^{a_{n,\theta}+n^{-\theta}+h} w_{\sigma,\theta}(y)\,\mathrm{d}y - \int_{a_{n,\theta}}^{a_{n,\theta}+n^{-\theta}} w_{\sigma,\theta}(\xi)\,\mathrm{d}\xi\right) \\ &= n^{-\theta(1/\varrho-1)} \left(h\,n^{-\theta\sigma} - \int_{a_{n,\theta}}^{a_{n,\theta}+h} w_{\sigma,\theta}(\xi)\,\mathrm{d}\xi\right) \\ &\leq h\,n^{-\theta(\sigma+1/\varrho-1)},\end{aligned}$$

i.e., $\|\Delta_h w_{\sigma,\theta} \,|\, L_\varrho(\mathscr{L}_{n,\theta})\|^\varrho \leq h^\varrho\, n^{-\theta(\sigma\varrho+1-\varrho)}$, as well as

$$\begin{aligned}\|\Delta_h w_{\sigma,\theta} \,|\, L_\varrho(\mathscr{T}_{n,\theta})\| &\leq \left(2n^{-\theta}\right)^{1/\varrho-1} \int_{a_{n,\theta}+n^{-\theta}}^{a_{n,\theta}+3\,n^{-\theta}} \big(w_{\sigma,\theta}(\xi+h) - w_{\sigma,\theta}(\xi)\big)\,\mathrm{d}\xi \\ &\lesssim n^{-\theta(1/\varrho-1)} \left(\int_{a_{n,\theta}+n^{-\theta}}^{a_{n,\theta}+3\,n^{-\theta}} w_{\sigma,\theta}(\xi)\,\mathrm{d}\xi - \int_{a_{n,\theta}+n^{-\theta}+h}^{a_{n,\theta}+3\,n^{-\theta}+h} w_{\sigma,\theta}(\xi)\,\mathrm{d}\xi\right) \\ &= n^{-\theta(1/\varrho-1)} \left(\int_{a_{n,\theta}+n^{-\theta}}^{a_{n,\theta}+n^{-\theta}+h} \underbrace{w_{\sigma,\theta}(\xi)}_{=n^{-\theta\sigma}}\,\mathrm{d}\xi - \int_{a_{n,\theta}+3\,n^{-\theta}}^{a_{n,\theta}+3\,n^{-\theta}+h} \underbrace{w_{\sigma,\theta}(\xi)}_{=0}\,\mathrm{d}\xi\right) \\ &= h\,n^{-\theta(\sigma+1/\varrho-1)}\end{aligned}$$

so that $\|\Delta_h w_{\sigma,\theta} \,|\, L_\varrho(\mathscr{T}_{n,\theta})\|^\varrho \lesssim h^\varrho \, n^{-\theta(\sigma\varrho+1-\varrho)}$. Further we have $\sigma\varrho + 1 - \varrho > 0$ because $1/\varrho \geq 1 \geq 1-\sigma$. So, (11) and $h \leq n^{-\theta}$ yield that also

$$\|\Delta_h w_{\sigma,\theta} \,|\, L_\varrho(\mathscr{R}_{n,\theta})\|^\varrho \sim h^{\sigma\varrho+1} = h^\varrho \, h^{\sigma\varrho+1-\varrho} \lesssim h^\varrho \, n^{-\theta(\sigma\varrho+1-\varrho)}.$$

Combining the latter estimates shows that

$$\left\| \Delta_h w_{\sigma,\theta} \,\middle|\, L_\varrho\left(\bigcup_{n=2}^{N(h,\theta)} (\mathscr{L}_{n,\theta} \cup \mathscr{T}_{n,\theta} \cup \mathscr{R}_{n,\theta}) \right) \right\|^\varrho \lesssim h^\varrho \sum_{n=2}^{N(h,\theta)} n^{-\theta(\sigma\varrho+1-\varrho)} \quad \text{for all} \quad 0 < \varrho \leq 1. \tag{12}$$

Now additionally assume $1/\varrho < (1-\sigma)/(1-1/\theta)$. Then there holds $0 < \theta(\sigma\varrho + 1 - \varrho) < 1$ and hence

$$\begin{aligned} \sum_{n=2}^{N(h,\theta)} n^{-\theta(\sigma\varrho+1-\varrho)} &\leq \int_1^{N(h,\theta)} x^{-\theta(\sigma\varrho+1-\varrho)} \, \mathrm{d}x \\ &= \frac{1}{1-\theta(\sigma\varrho+1-\varrho)} \left(N(h,\theta)^{1-\theta(\sigma\varrho+1-\varrho)} - 1 \right), \end{aligned}$$

where

$$\begin{aligned} N(h,\theta)^{1-\theta(\sigma\varrho+1-\varrho)} - 1 &\leq \left(\frac{1}{3} h^{-1/\theta} + 1 \right)^{1-\theta(\sigma\varrho+1-\varrho)} - 1 \\ &\leq \left(\frac{1}{3} h^{-1/\theta} \right)^{1-\theta(\sigma\varrho+1-\varrho)} \\ &\sim h^{-\varrho+\sigma\varrho+1-1/\theta}. \end{aligned}$$

Therefore, we arrive at

$$\left\| \Delta_h w_{\sigma,\theta} \,\middle|\, L_\varrho\left(\bigcup_{n=2}^{N(h,\theta)} (\mathscr{L}_{n,\theta} \cup \mathscr{T}_{n,\theta} \cup \mathscr{R}_{n,\theta}) \right) \right\|^\varrho \lesssim h^{\sigma\varrho+1-1/\theta}.$$

Moreover, (10) and $\theta > 1$ yield that also

$$\begin{aligned} &\left\| \Delta_h w_{\sigma,\theta} \,\middle|\, L_\varrho\left(\bigcup_{n=N(h,\theta)+1}^{\infty} (\mathscr{L}_{n,\theta} \cup \mathscr{T}_{n,\theta} \cup \mathscr{R}_{n,\theta}) \right) \right\|^\varrho \\ &\qquad = \sum_{n=N(h,\theta)+1}^{\infty} \left\| \Delta_h w_{\sigma,\theta} \,\middle|\, L_\varrho\big([a_{n,\theta}, a_{n,\theta} + 4\, n^{-\theta}]\big) \right\|^\varrho \end{aligned}$$

is bounded by

$$\sum_{n=N(h,\theta)+1}^{\infty} 4\,n^{-\theta}\,(2\,h^{\sigma})^{\varrho} \lesssim h^{\sigma\varrho} \int_{N(h,\theta)}^{\infty} x^{-\theta}\,\mathrm{d}x$$
$$= h^{\sigma\varrho}\,\frac{1}{\theta-1}\,N(h,\theta)^{1-\theta}$$
$$\lesssim h^{\sigma\varrho+1-1/\theta}.$$

Finally, we clearly have $w_{\sigma,\theta}(x) = 0$ on $\mathscr{L}_\theta \cup \mathscr{R}_\theta$ and hence $\|\Delta_h w_{\sigma,\theta} \,|\, L_\varrho(\mathscr{R}_\theta)\| = 0$, as well as

$$\|\Delta_h w_{\sigma,\theta} \,|\, L_\varrho(\mathscr{L}_\theta)\|^{\varrho} = \int_{a_{2,\theta}}^{a_{2,\theta}+h} w_{\sigma,\theta}(y)^{\varrho}\,\mathrm{d}y$$
$$\leq \int_0^h x^{\sigma\varrho}\,\mathrm{d}x$$
$$= \frac{1}{\sigma\varrho+1}\,h^{\sigma\varrho+1}$$
$$\lesssim h^{\sigma\varrho+1-1/\theta}.$$

Altogether, this shows (9) and thus the proof is complete. ■

Remark 2.4. As already mentioned, the piecewise construction of $w_{\sigma,\theta}$ in (5) goes back to Simon. In [22, Sect. 4] he used first order differences to show that for fixed σ there exists some θ such that $w_{\sigma,\theta}$ satisfies certain Sobolev regularity assertions *independently* of the integrability parameter ϱ (restricted to $[1,\infty]$) of the spaces. However, for the application we have in mind, stronger results in *Besov* spaces are needed which required a more detailed analysis. Note that indeed our characterizations in Lemma 2.3 reveal a fairly complicated interplay of the parameters of $w_{\sigma,\theta}$ and its smoothness and integrability which is not visible in the results of Simon.

Further, we like to stress that some parameter restrictions in Lemma 2.3(iii) are stronger than required. If $\varrho = \infty$, our proof actually works for all $0 < \sigma < 1 < \theta < \infty$. Moreover, the upper bound on $1/\varrho$ in (7) seems to be an artifact of our proof technique. At least for the "only if" part it can be dropped, as can be seen easily using complex interpolation.

In order to proceed, again let $\sigma \in \mathbb{R}_+$ and $1 < \theta < \infty$. Then, based on $w_{\sigma,\theta}$ as defined in (5) above, let us set

$$\begin{aligned} v_{\sigma,\theta}\colon \mathbb{R}_+ \to \mathbb{R}, \qquad & r \mapsto v_{\sigma,\theta}(r) := w_{\sigma,\theta}\big(16\,\zeta(\theta)\,r - 4\,\zeta(\theta)\big) - w_{\sigma,\theta}\big(16\,\zeta(\theta)\,r - 8\,\zeta(\theta)\big),\\ u_{\sigma,\theta}\colon \mathbb{R}_+ \to \mathbb{R}, \qquad & r \mapsto u_{\sigma,\theta}(r) := \int_0^r v_{\sigma,\theta}(\xi)\,\mathrm{d}\xi. \end{aligned} \tag{13}$$

Lemma 2.5 (Properties of $v_{\sigma,\theta}$ and $u_{\sigma,\theta}$)**.** *Let $\sigma \in \mathbb{R}_+$ and $1 < \theta < \infty$. Then*

(i) the supports of $u_{\sigma,\theta}$ and $v_{\sigma,\theta}$ are contained in $\widetilde{\mathscr{S}_\theta} := [1/4, 3/4]$.

(ii) $v_{\gamma\sigma,\theta} = |v_{\sigma,\theta}|^{\gamma-1} v_{\sigma,\theta}$ for all $\gamma \in \mathbb{R}_+$.

(iii) $u_{\sigma,\theta} \in C^1(\mathbb{R}_+)$ with $u'_{\sigma,\theta} = v_{\sigma,\theta}$.

(iv) for all $0 < \varrho, q \leq \infty$ and $s < 1$ we have

$$u_{\sigma,\theta} \in B^{1+s}_{\varrho,q}(\mathbb{R}_+) \qquad \text{if and only if} \qquad w_{\sigma,\theta} \in B^s_{\varrho,q}(\mathbb{R}).$$

(v) for all $1 < \varrho < \infty$ we have

$$u_{\sigma,\theta} \in W^2_\varrho(\mathbb{R}_+) \qquad \text{if and only if} \qquad w_{\sigma,\theta} \in W^1_\varrho(\mathbb{R}).$$

Proof. We use $\operatorname{supp}(w_{\sigma,\theta}) \subseteq \mathscr{S}_\theta = [a_{2,\theta}, a_{\infty,\theta}] = [4, 4\,\zeta(\theta)]$, as shown in Lemma 2.1(i), to deduce the representation

$$v_{\sigma,\theta}(r) = \begin{cases} w_{\sigma,\theta}(t) & \text{if} \quad r = \dfrac{t + 4\,\zeta(\theta)}{16\,\zeta(\theta)} \in \left[\dfrac{1}{4} + \dfrac{1}{4\,\zeta(\theta)}, \dfrac{1}{2}\right], \\ -w_{\sigma,\theta}(t') & \text{if} \quad r = \dfrac{t' + 8\,\zeta(\theta)}{16\,\zeta(\theta)} \in \left[\dfrac{1}{2} + \dfrac{1}{4\,\zeta(\theta)}, \dfrac{3}{4}\right], \\ 0 & \text{else}. \end{cases} \tag{14}$$

This proves (i) for $v_{\sigma,\theta}$. Moreover, for $0 < r < 1/4$ we have $u_{\sigma,\theta}(r) = \int_0^r 0 \,\mathrm{d}\xi = 0$, while for $r > 3/4$ we may write

$$u_{\sigma,\theta}(r) = \int_0^r v_{\sigma,\theta}(\xi)\,\mathrm{d}\xi = 0 + \int_{1/4}^{1/2} v_{\sigma,\theta}(\xi)\,\mathrm{d}\xi - \int_{1/2}^{3/4} -v_{\sigma,\theta}(\xi)\,\mathrm{d}\xi + 0 = 0$$

which shows (i) for $u_{\sigma,\theta}$. Further, (ii) directly follows from (14) and Lemma 2.1(iii).

We are left with proving the regularity assertions (iii)–(v). The fact that $u_{\sigma,\theta} \in C^1(\mathbb{R}_+)$ with $u'_{\sigma,\theta} = v_{\sigma,\theta}$ is a consequence of the fundamental theorem of calculus and the continuity of $v_{\sigma,\theta}$, cf. (13) and Lemma 2.1(i) again. This shows (iii).

If we assume that $w_{\sigma,\theta} \in B^s_{\varrho,q}(\mathbb{R})$, then also

$$\widetilde{v_{\sigma,\theta}} := w_{\sigma,\theta}\big(16\,\zeta(\theta)\,\cdot - 4\,\zeta(\theta)\big) - w_{\sigma,\theta}\big(16\,\zeta(\theta)\,\cdot - 8\,\zeta(\theta)\big) \in B^s_{\varrho,q}(\mathbb{R})$$

because $B^s_{\varrho,q}(\mathbb{R})$ is invariant under diffeomorphic coordinate transformations; see, e.g., Triebel [23, Sect. 2.10.2]. Since $v_{\sigma,\theta} = \widetilde{v_{\sigma,\theta}}\big|_{\mathbb{R}_+}$ this yields $v_{\sigma,\theta} \in B^s_{\varrho,q}(\mathbb{R}_+)$. On the other hand,

$v_{\sigma,\theta} \in B^s_{\varrho,q}(\mathbb{R}_+)$ implies that there exists $g \in B^s_{\varrho,q}(\mathbb{R})$ such that $v_{\sigma,\theta} = g\big|_{\mathbb{R}_+}$. Now let $\chi \in C^\infty(\mathbb{R})$ with

$$\chi(x) = \begin{cases} 1 & \text{for} \quad \frac{1}{4} \leq x \leq \frac{1}{2} + \frac{1}{6\,\zeta(\theta)}, \\ 0 & \text{for} \quad x \leq \frac{1}{5} \quad \text{or} \quad \frac{1}{2} + \frac{1}{5\,\zeta(\theta)} \leq x. \end{cases}$$

Then, according to a multiplication theorem by Triebel [24, Sect. 4.2.2], we conclude that $w_{\sigma,\theta} = \chi\, g \in B^s_{\varrho,q}(\mathbb{R})$. Due to (iii), this shows that for all $0 < \varrho, q \leq \infty$ and $s \in \mathbb{R}$

$$u'_{\sigma,\theta} \in B^s_{\varrho,q}(\mathbb{R}_+) \qquad \text{if and only if} \qquad w_{\sigma,\theta} \in B^s_{\varrho,q}(\mathbb{R}). \tag{15}$$

In addition, we may extend $u_{\sigma,\theta} \in C^1(\mathbb{R}_+)$ by zero in order to obtain $\widetilde{u_{\sigma,\theta}} \in C^1(\mathbb{R})$. Using the characterization of Besov spaces in terms of first order differences (cf. Remark 1.1(i)), we see that this gives $\widetilde{u_{\sigma,\theta}} \in B^{1-\varepsilon}_{\infty,q}(\mathbb{R})$ for all $0 < \varepsilon < 1$. Choosing ε small enough such that $s < 1 - \varepsilon$ then shows $\widetilde{u_{\sigma,\theta}} \in B^s_{\infty,q}(\mathbb{R}) \hookrightarrow B^s_{\varrho,q}(\mathbb{R})$, i.e., $u_{\sigma,\theta} \in B^s_{\varrho,q}(\mathbb{R}_+)$, where we used Proposition 2.2 and the compact support of $\widetilde{u_{\sigma,\theta}}$. Therefore, (iv) follows from Proposition 2.2(i).

Since Sobolev spaces W^k_ϱ can be identified with special Triebel-Lizorkin spaces $F^k_{\varrho,2}$, we can argue similarly for this case. Instead of (15) we now have that for every $k \in \mathbb{N}_0$ (particularly for $k = 1$) and $1 < \varrho < \infty$ there holds $u'_{\sigma,\theta} \in W^k_\varrho(\mathbb{R}_+)$ if and only if $w_{\sigma,\theta} \in W^k_\varrho(\mathbb{R})$. Further, from (i) and (iii) we clearly have $u_{\sigma,\theta} \in W^1_\varrho(\mathbb{R}_+)$. Together this shows (v) and hence the proof is complete. ■

Next, for $d \in \mathbb{N}$ let $\Omega \subseteq \mathbb{R}^d$ be either $\mathbb{R}^d$ itself, a bounded Lipschitz domain, or an interval (if $d = 1$), and assume that Ω contains the Euclidean unit ball

$$B_1(0) := \{x \in \mathbb{R}^d \mid |x|_2 < 1\}.$$

Given $\sigma \in \mathbb{R}_+$, as well as $1 < p, \theta < \infty$, for all test functions $\psi \in \mathcal{D}(\Omega)$ we then let

$$u_{\sigma,\theta,d}(\psi) := \int_\Omega u_{\sigma,\theta}(|x|_2)\, \psi(x)\, \mathrm{d}x \quad \text{and} \quad f^{[p]}_{\sigma,\theta,d}(\psi) := \int_\Omega v_{(p-1)\sigma,\theta}(|x|_2) \left\langle \frac{x}{|x|_2}, \nabla\psi(x) \right\rangle_{\mathbb{R}^d} \mathrm{d}x. \tag{16}$$

Since Lemma 2.5(iii) implies $u_{\sigma,\theta}, v_{(p-1)\sigma,\theta} \in L_\infty(\mathbb{R}_+)$, it is easily seen that these integrals are bounded linear functionals of ψ such that we actually deal with distributions $u_{\sigma,\theta,d}$ and $f^{[p]}_{\sigma,\theta,d}$ from $\mathcal{D}'(\Omega)$. Moreover, these functionals are closely related:

Lemma 2.6. *Let $d \in \mathbb{N}$ and $1 < p, \theta < \infty$, as well as $\sigma \in \mathbb{R}_+$. Further let $\Omega \subseteq \mathbb{R}^d$ be either $\mathbb{R}^d$ itself, a bounded Lipschitz domain, or an interval (if $d = 1$), and assume $B_1(0) \subseteq \Omega$. Then*

(i) we have

$$\mathrm{supp}(u_{\sigma,\theta,d}), \mathrm{supp}(f^{[p]}_{\sigma,\theta,d}) \subseteq B_{4/5}(0),$$

(ii) for all $x \in \Omega$ it holds

$$A(\nabla u_{\sigma,\theta,d})(x) = v_{(p-1)\sigma,\theta}(|x|_2)\,\frac{x}{|x|_2} = \nabla u_{(p-1)\sigma,\theta,d}(x), \tag{17}$$

(iii) there holds $f^{[p]}_{\sigma,\theta,d} \in W^{-1}_{p'}(\Omega)$ and $u_{\sigma,\theta,d} \in W^1_p(\Omega)$ constitutes a weak solution u to

$$-\Delta_p u = f^{[p]}_{\sigma,\theta,d} \qquad \textit{and} \qquad u|_{\partial\Omega} = 0.$$

Proof. In view of Lemma 2.5 assertion (i) is obvious. Further, it is clear that the distribution $u_{\sigma,\theta,d} \in \mathcal{D}'(\Omega)$ is regular and can be identified with the function $\big(u_{\sigma,\theta} \circ r_d\big)\big|_\Omega \in C^1(\Omega)$, where we set

$$r_d \colon \mathbb{R}^d \to \mathbb{R}, \qquad x \mapsto r_d(x) := |x|_2\,. \tag{18}$$

With this interpretation we have

$$\frac{\partial u_{\sigma,\theta,d}}{\partial x_j}(x) = 0 = v_{\sigma,\theta}(|x|_2)\,\frac{x_j}{|x|_2}, \qquad j = 1, \ldots, d,$$

for all $x \in B_{1/4}(0)$, while on $\Omega \setminus B_{1/8}(0)$ the chain rule and Lemma 2.5(iii) give

$$\frac{\partial u_{\sigma,\theta,d}}{\partial x_j} = \frac{\partial (u_{\sigma,\theta} \circ r_d)}{\partial x_j} = v_{\sigma,\theta}(r_d(\cdot))\,\frac{\partial r_d}{\partial x_j},$$

where

$$\frac{\partial r_d}{\partial x_j}(x) = \frac{\partial}{\partial x_j}\left[\left(\sum_{k=1}^d x_k^2\right)^{1/2}\right](x) = \frac{1}{2}\left(\sum_{k=1}^d x_k^2\right)^{-1/2} 2x_j = \frac{x_j}{|x|_2}.$$

Together this shows

$$\frac{\partial u_{\sigma,\theta,d}}{\partial x_j}(x) = v_{\sigma,\theta}(|x|_2)\,\frac{x_j}{|x|_2} \qquad \text{for all} \qquad j = 1, \ldots, d \quad \text{and} \quad x \in \Omega \tag{19}$$

and hence

$$\left\| u_{\sigma,\theta,d} \,\middle|\, W_p^1(\Omega) \right\| = \| u_{\sigma,\theta,d} \,|\, L_p(\Omega) \| + \sum_{j=1}^d \left\| \frac{\partial u_{\sigma,\theta,d}}{\partial x_j} \,\middle|\, L_p(\Omega) \right\| < \infty$$

since $u_{\sigma,\theta}, v_{\sigma,\theta} \in L_\infty(\mathbb{R}_+)$ with compact support and $|x_j/\left|x\right|_2| \leq 1$. So, we can conclude $u_{\sigma,\theta,d} \in W_{p,0}^1(\Omega)$. Further, as a direct consequence of (19), we obtain

$$\left|\nabla u_{\sigma,\theta,d}(x)\right|_2 = \left(\sum_{j=1}^d \left| \frac{\partial u_{\sigma,\theta,d}}{\partial x_j}(x) \right|^2 \right)^{1/2} \overset{(19)}{=} \left(\frac{\left|v_{\sigma,\theta}(\left|x\right|_2)\right|^2}{\left|x\right|_2^2} \sum_{j=1}^d \left|x_j\right|^2 \right)^{1/2} = \left|v_{\sigma,\theta}(\left|x\right|_2)\right|$$

such that by Lemma 2.5(ii) with $\gamma := p-1$ we have that

$$\begin{aligned} A(\nabla u_{\sigma,\theta,d})(x) &= \left|\nabla u_{\sigma,\theta,d}(x)\right|_2^{p-2} \, \nabla u_{\sigma,\theta,d}(x) \\ &= \left|v_{\sigma,\theta}(\left|x\right|_2)\right|^{(p-1)-1} \, v_{\sigma,\theta}(\left|x\right|_2) \, \frac{x}{\left|x\right|_2} \\ &= v_{(p-1)\sigma,\theta}(\left|x\right|_2) \, \frac{x}{\left|x\right|_2} \end{aligned}$$

holds for all $x \in \Omega$. Together with (19) this proves (17), as well as

$$\int_\Omega \langle A(\nabla u_{\sigma,\theta,d})(x), \nabla\psi(x) \rangle_{\mathbb{R}^d} \, \mathrm{d}x = \int_\Omega v_{(p-1)\sigma,\theta}(\left|x\right|_2) \left\langle \frac{x}{\left|x\right|_2}, \nabla\psi(x) \right\rangle_{\mathbb{R}^d} \mathrm{d}x = f_{\sigma,\theta,d}^{[p]}(\psi) \quad (20)$$

for each $\psi \in \mathcal{D}(\Omega)$. In other words, there holds $-\Delta_p(u_{\sigma,\theta,d}) = f_{\sigma,\theta,d}^{[p]}$ in the weak sense. Finally, Hölder's inequality on $B_1(0) \subseteq \Omega$ proves

$$\begin{aligned} \left| f_{\sigma,\theta,d}^{[p]}(\psi) \right| &\leq \int_\Omega \left| v_{(p-1)\sigma,\theta}(\left|x\right|_2) \right| \, \left|\nabla\psi(x)\right|_2 \, \mathrm{d}x \\ &\leq \left\| v_{(p-1)\sigma,\theta} \,\middle|\, L_\infty(\mathbb{R}_+) \right\| \int_{B_1(0)} \left|\nabla\psi(x)\right|_2 \, \mathrm{d}x \\ &\lesssim \left\| v_{(p-1)\sigma,\theta} \,\middle|\, L_\infty(\mathbb{R}_+) \right\| \, \left\| \psi \,\middle|\, W_p^1(\Omega) \right\|, \qquad \psi \in \mathcal{D}(\Omega). \end{aligned}$$

Since by definition $\mathcal{D}(\Omega)$ is dense $W_{p,0}^1(\Omega)$ we therefore have $f_{\sigma,\theta,d}^{[p]} \in (W_{p,0}^1(\Omega))' = W_{p'}^{-1}(\Omega)$ and the proof is complete. ■

In order to provide further regularity assertions for $u_{\sigma,\theta,d}$ and $f_{\sigma,\theta,d}^{[p]}$, we will need the subsequent result which characterizes the smoothness and integrability of rotationally invariant functions. Therein r_d has the same meaning as in (18).

Proposition 2.7. *Let $0 < a < b < \infty$. Assume that $g\colon \mathbb{R}_+ \to \mathbb{C}$ is measurable with support* $\operatorname{supp}(g) \subseteq [a,b]$ *and let $g_d := g \circ r_d$. Then*

(i) $g_d\colon \mathbb{R}^d \to \mathbb{C}$ *is well-defined almost everywhere.*

(ii) for $0 < \varrho, q \leq \infty$ and $s > d \max\{0, 1/\varrho - 1\}$ there holds

$$g_d \in L_\varrho(\mathbb{R}^d) \qquad \textit{if and only if} \qquad g \in L_\varrho(\mathbb{R}_+), \tag{21}$$

as well as

$$g_d \in B^s_{\varrho,q}(\mathbb{R}^d) \qquad \textit{if and only if} \qquad g \in B^s_{\varrho,q}(\mathbb{R}_+).$$

(iii) for $1 < \varrho < \infty$ and $k \in \mathbb{N}$ there holds

$$g_d \in W^k_\varrho(\mathbb{R}^d) \qquad \textit{if and only if} \qquad g \in W^k_\varrho(\mathbb{R}_+).$$

Proof. With g and r_d also g_d is measurable such that it can be represented as an almost everywhere convergent pointwise limit of simple functions. This proves (i).

If $d = 1$, the equivalences in (ii) and (iii) are trivial, as g vanishes in a neighborhood of the origin. So let us assume $d \geq 2$. Then for $\varrho < \infty$ the first assertion in (ii) follows from a simple transformation into (generalized) polar coordinates $x = r\,\vartheta(\phi)$ with $(r,\phi) \in [0,\infty) \times \Phi$:

$$\left\| g_d \,\middle|\, L_\varrho(\mathbb{R}^d) \right\|^\varrho = \int_{\mathbb{R}^d} |g(r_d(x))|^\varrho \,\mathrm{d}x = \int_\Phi \int_a^b |g(r)|^\varrho \, r^{d-1} T(\phi) \,\mathrm{d}r \,\mathrm{d}\phi \sim \| g \,|\, L_\varrho((0,\infty)) \|^\varrho ,$$

where we used that $\operatorname{supp}(g) \subseteq [a,b]$ and that T is some tensor product of trigonometric functions defined on $\Phi \subset [-\pi,\pi]^{d-1}$. For $\varrho = \infty$ the equivalence (21) is obvious.

It remains to prove the equivalences for multivariate Besov and Sobolev spaces. In case of $B^s_{\varrho,q}$ and $\varrho = \infty$, this follows from results due to Sickel *et al.* [20, Thm. 2], while the case $0 < \varrho < \infty$ is covered by [20, Cor. 1 & 2]. However, [20, Cor. 1 & 2] also covers the assertion for Sobolev spaces since $W^k_\varrho = F^k_{\varrho,2}$ if $1 < \varrho < \infty$. ■

Lemma 2.8 (Regularity of $u_{\sigma,\theta,d}$, $A(\nabla u_{\sigma,\theta,d})$, and $f^{[p]}_{\sigma,\theta,d}$)**.** *Let $d \in \mathbb{N}$ and $1 < p, \theta < \infty$, as well as $\sigma \in \mathbb{R}_+$. Further let $\Omega \subseteq \mathbb{R}^d$ be either $\mathbb{R}^d$ itself, a bounded Lipschitz domain, or an interval (if $d = 1$), and assume $B_1(0) \subseteq \Omega$. Moreover, let $0 < \varrho, q \leq \infty$ and $s \in \mathbb{R}$ with $d \max\{0, 1/\varrho - 1\} < s < 1$. Then*

(i) there holds

$$u_{\sigma,\theta,d} \in B^{1+s}_{\varrho,q}(\Omega) \qquad \textit{if and only if} \qquad w_{\sigma,\theta} \in B^s_{\varrho,q}(\mathbb{R})$$

and $1 < \varrho < \infty$ implies that

$$u_{\sigma,\theta,d} \in W^2_\varrho(\Omega) \qquad \textit{if and only if} \qquad w_{\sigma,\theta} \in W^1_\varrho(\mathbb{R}).$$

(ii) we have

$$A(\nabla u_{\sigma,\theta,d}) \in \big(B^s_{\varrho,q}(\Omega)\big)^d \qquad \text{if and only if} \qquad w_{(p-1)\sigma,\theta} \in B^s_{\varrho,q}(\mathbb{R})$$

and $1<\varrho<\infty$ *implies that*

$$A(\nabla u_{\sigma,\theta,d}) \in \big(W^1_{\varrho}(\Omega)\big)^d \qquad \text{if and only if} \qquad w_{(p-1)\sigma,\theta} \in W^1_{\varrho}(\mathbb{R}).$$

Additionally assume that $\min\{\varrho, q\} > 1$. *Then*

(iii) $A(\nabla u_{\sigma,\theta,d}) \in \big(B^s_{\varrho,q}(\Omega)\big)^d$ *implies* $f^{[p]}_{\sigma,\theta,d} \in B^{-1+s}_{\varrho,q}(\Omega)$,

(iv) $w'_{(p-1)\sigma,\theta} \in L_\varrho(\mathbb{R})$ *implies* $f^{[p]}_{\sigma,\theta,d} \in L_\varrho(\Omega)$.

Proof. Recall that $u_{\sigma,\theta,d} \in \mathcal{D}'(\Omega)$ can be identified with the function $u_{\sigma,\theta} \circ r_d$ restricted to Ω. Hence, $u_{\sigma,\theta} \circ r_d \in B^{1+s}_{\varrho,q}(\mathbb{R}^d)$ implies $u_{\sigma,\theta,d} \in B^{1+s}_{\varrho,q}(\Omega)$. On the other hand, if $u_{\sigma,\theta,d} \in B^{1+s}_{\varrho,q}(\Omega)$, then by definition there exists $\widetilde{u} \in B^{1+s}_{\varrho,q}(\mathbb{R}^d)$ with $\widetilde{u}\big|_\Omega = u_{\sigma,\theta,d}$. Now let $\chi \in C^\infty(\mathbb{R}^d)$ with

$$\chi(x) = \begin{cases} 1 & \text{if} \quad x \in B_{4/5}(0), \\ 0 & \text{if} \quad x \in \mathbb{R}^d \setminus B_1(0). \end{cases}$$

Then $u_{\sigma,\theta} \circ r_d = \chi\,\widetilde{u} \in B^{1+s}_{\varrho,q}(\mathbb{R}^d)$ due to [24, Sect. 4.2.2]. Therefore, $u_{\sigma,\theta,d} \in B^{1+s}_{\varrho,q}(\Omega)$ is equivalent to $u_{\sigma,\theta} \circ r_d \in B^{1+s}_{\varrho,q}(\mathbb{R}^d)$. By Lemma 2.5(i) and Proposition 2.7(ii) this holds if and only if $u_{\sigma,\theta} \in B^{1+s}_{\varrho,q}(\mathbb{R}_+)$. Since we assume $s<1$, we can use Lemma 2.5(iv) to see that this in turn is equivalent to $w_{\sigma,\theta} \in B^s_{\varrho,q}(\mathbb{R})$. Thus, we have shown (i) in the case of Besov spaces. For Sobolev spaces we can argue similarly.

Next we apply (i) to deduce that $w_{(p-1)\sigma,\theta} \in B^s_{\varrho,q}(\mathbb{R})$ is equivalent to $u_{(p-1)\sigma,\theta,d} \in B^{1+s}_{\varrho,q}(\Omega)$. By Proposition 2.2(i) this holds if and only if $u_{(p-1)\sigma,\theta,d} \in B^s_{\varrho,q}(\Omega)$ and $\nabla u_{(p-1)\sigma,\theta,d} \in (B^s_{\varrho,q}(\Omega))^d$. Since we assume that $s<1$, the first condition is always fulfilled (cf. the proof of Lemma 2.5!), and by Lemma 2.6(ii) $\nabla u_{(p-1)\sigma,\theta,d}$ is nothing but $A(\nabla u_{\sigma,\theta,d})$. Also here the proof for Sobolev spaces is essentially the same.

Let us prove (iii). To this end, we note that $A(\nabla u_{\sigma,\theta,d}) \in \big(B^s_{\varrho,q}(\Omega)\big)^d$ implies that for every $j = 1, \ldots, d$ we have

$$A(\nabla u_{\sigma,\theta,d})_j \in \widetilde{B}^s_{\varrho,q}(\Omega) := \Big\{ g \in \mathcal{D}'(\Omega) \,\Big|\, \exists\, \widetilde{g} \in B^s_{\varrho,q}(\mathbb{R}^d) \text{ with } g = \widetilde{g}\big|_\Omega \text{ and } \operatorname{supp}(\widetilde{g}) \subseteq \overline{\Omega} \Big\}$$

according to Lemma 2.6, where we set

$$\Big\| g \,\Big|\, \widetilde{B}^s_{\varrho,q}(\Omega) \Big\| := \inf_{\substack{\widetilde{g} \in B^s_{\varrho,q}(\mathbb{R}^d) \colon g = \widetilde{g}|_\Omega, \\ \operatorname{supp}(\widetilde{g}) \subseteq \overline{\Omega}}} \big\| \widetilde{g} \,\big|\, B^s_{\varrho,q}(\mathbb{R}^d) \big\|,$$

cf. Triebel [26, Def. 2.1]. Further from [26, Thm. 3.30] and $\min\{\varrho, q\} > 1$ it follows that $\widetilde{B}^s_{\varrho,q}(\Omega) = (B^{-s}_{\varrho',q'}(\Omega))'$ such that for all $j = 1, \ldots, d$ we can estimate

$$|A(\nabla u_{\sigma,\theta,d})_j(\varphi)| \lesssim \left\| \varphi \,\middle|\, B^{-s}_{\varrho',q'}(\Omega) \right\|, \qquad \varphi \in \mathcal{D}(\Omega).$$

Therefore the representation formula (20) together with Proposition 2.2(i) yield that for all $\psi \in \mathcal{D}(\Omega)$ there holds

$$\begin{aligned} \left| f^{[p]}_{\sigma,\theta,d}(\psi) \right| &= \left| \sum_{j=1}^{d} \int_\Omega A(\nabla u_{\sigma,\theta,d})_j(x) \, \frac{\partial \psi}{\partial x_j}(x) \, \mathrm{d}x \right| \\ &\le \sum_{j=1}^{d} \left| A(\nabla u_{\sigma,\theta,d})_j \left(\frac{\partial \psi}{\partial x_j} \right) \right| \\ &\lesssim \sum_{j=1}^{d} \left\| \frac{\partial \psi}{\partial x_j} \,\middle|\, B^{-s}_{\varrho',q'}(\Omega) \right\| \\ &\lesssim \left\| \psi \,\middle|\, B^{1-s}_{\varrho',q'}(\Omega) \right\| \\ &\le \left\| \psi \,\middle|\, \widetilde{B}^{1-s}_{\varrho',q'}(\Omega) \right\|. \end{aligned}$$

Now we again employ [26, Thm. 3.30] to see that $\mathcal{D}(\Omega)$ is dense in $\widetilde{B}^{1-s}_{\varrho',q'}(\Omega)$ and hence we conclude

$$f^{[p]}_{\sigma,\theta,d} \in \left(\widetilde{B}^{1-s}_{\varrho',q'}(\Omega) \right)' = B^{-1+s}_{\varrho,q}(\Omega),$$

as claimed.

It remains to prove (iv). For this purpose, note that for $f^{[p]}_{\sigma,\theta,d} \in L_\varrho(\Omega) = (L_{\varrho'}(\Omega))'$ it is sufficient to find $f_d \in L_\varrho(\Omega)$ such that

$$f^{[p]}_{\sigma,\theta,d}(\psi) = \int_\Omega f_d(x) \, \psi(x) \, \mathrm{d}x, \qquad \psi \in \mathcal{D}(\Omega), \tag{22}$$

since $\mathcal{D}(\Omega)$ is dense in $L_{\varrho'}(\Omega)$ if $1 \le \varrho' < \infty$. We claim that this f_d is given by the restriction of $f_d = f \circ r_d$ to Ω, where

$$f(r) := -v'_{(p-1)\sigma,\theta}(r) - v_{(p-1)\sigma,\theta}(r) \, \frac{d-1}{r} \qquad \text{for a.e.} \qquad r > 0.$$

Recall that, due to Lemma 2.5, $v_{(p-1)\sigma,\theta}$ is continuous with support in the interval $[1/4, 3/4]$. Hence, the function $r \mapsto g(r) := v_{(p-1)\sigma,\theta}(r)/r$ belongs to $L_\varrho(\mathbb{R}_+)$. Moreover, it is clear that our assumption $w'_{(p-1)\sigma,\theta} \in L_\varrho(\mathbb{R})$ implies that also $v'_{(p-1)\sigma,\theta} \in L_\varrho(\mathbb{R}_+)$. Therefore, we have

$f \in L_\varrho(\mathbb{R}_+)$ and from Proposition 2.7 we conclude $f_d \in L_\varrho(\mathbb{R}^d)$. This shows that indeed $f_d \in L_\varrho(\Omega)$. Thus, we are left with proving (22). To this end, let $\psi \in \mathcal{D}(\Omega)$ be arbitrarily fixed and assume for a moment that $d \geq 2$. Then $B_1(0) \subseteq \Omega$ and a transformation into polar coordinates $x = r\,\vartheta(\phi)$ with $(r, \phi) \in [0, \infty) \times \Phi$, yields

$$\int_\Omega f_d(x)\,\psi(x)\,\mathrm{d}x = \int_\Phi \int_{1/4}^{3/4} f(r)\,\psi(r\,\vartheta(\phi))\,r^{d-1}\,T(\phi)\,\mathrm{d}r\,\mathrm{d}\phi \tag{23}$$

(cf. the proof of Proposition 2.7). Since $v'_{(p-1)\sigma,\theta}$ belongs to $L_1(\mathbb{R}_+)$ and ψ is smooth, we may use integration by parts to see that

$$\begin{aligned}&\int_{1/4}^{3/4} v'_{(p-1)\sigma,\theta}(r)\,\psi(r\,\vartheta(\phi))\,r^{d-1}\,\mathrm{d}r \\ &\qquad = \left[v_{(p-1)\sigma,\theta}(r)\,\psi(r\,\vartheta(\phi))\,r^{d-1}\right]_{r=1/4}^{3/4} - \int_{1/4}^{3/4} v_{(p-1)\sigma,\theta}(r)\,\frac{\mathrm{d}}{\mathrm{d}r}\big(\psi(r\,\vartheta(\phi))\,r^{d-1}\big)\,(r)\,\mathrm{d}r\end{aligned}$$

is finite, because the boundary term vanishes and

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}r}\big(\psi(r\,\vartheta(\phi))\,r^{d-1}\big)\,(r) &= \frac{\mathrm{d}}{\mathrm{d}r}\,(\psi(r\,\vartheta(\phi)))\,(r)\,r^{d-1} + (d-1)\,\psi(r\,\vartheta(\phi))\,r^{d-2} \\ &= \langle \nabla\psi(r\,\vartheta(\phi)), \vartheta(\phi)\rangle_{\mathbb{R}^d}\;r^{d-1} + \frac{d-1}{r}\,\psi(r\,\vartheta(\phi))\,r^{d-1}\end{aligned}$$

as well as $v_{(p-1)\sigma,\theta}$ are bounded on $[1/4, 3/4]$. Hence, for the inner integral in (23) we find

$$\begin{aligned}&\int_{1/4}^{3/4} f(r)\,\psi(r\,\vartheta(\phi))\,r^{d-1}\,\mathrm{d}r \\ &\qquad = -\int_{1/4}^{3/4} v'_{(p-1)\sigma,\theta}(r)\,\psi(r\,\vartheta(\phi))\,r^{d-1}\,\mathrm{d}r - \int_{1/4}^{3/4} v_{(p-1)\sigma,\theta}(r)\,\frac{d-1}{r}\,\psi(r\,\vartheta(\phi))\,r^{d-1}\,\mathrm{d}r \\ &\qquad = \int_{1/4}^{3/4} v_{(p-1)\sigma,\theta}(r)\,\left\langle \nabla\psi(r\,\vartheta(\phi)), \frac{r\,\vartheta(\phi)}{r}\right\rangle_{\mathbb{R}^d} r^{d-1}\,\mathrm{d}r\end{aligned}$$

and thus (20) shows that we indeed have (22):

$$\begin{aligned}\int_\Omega f_d(x)\,\psi(x)\,\mathrm{d}x &= \int_\Phi \int_{1/4}^{3/4} f(r)\,\psi(r\,\vartheta(\phi))\,r^{d-1}\,T(\phi)\,\mathrm{d}r\,\mathrm{d}\phi \\ &= \int_\Phi \int_{1/4}^{3/4} v_{(p-1)\sigma,\theta}(r)\,\left\langle \nabla\psi(r\,\vartheta(\phi)), \frac{r\,\vartheta(\phi)}{r} \right\rangle_{\mathbb{R}^d} r^{d-1}\,T(\phi)\,\mathrm{d}r\,\mathrm{d}\phi \\ &= \int_\Omega v_{(p-1)\sigma,\theta,d}(x)\,\left\langle \nabla\psi(x), \frac{x}{|x|_2} \right\rangle_{\mathbb{R}^d} \mathrm{d}x \qquad (24) \\ &= f^{[p]}_{\sigma,\theta,d}(\psi).\end{aligned}$$

Finally, a similar calculation shows that (24) remains valid also for $d = 1$. So, the proof is complete. ■

Now we are well-prepared to give profound proofs of our main results stated in Theorems 1.3 and 1.6.

2.2 Proof of Theorem 1.3

Proof. Let $d \in \mathbb{N}$, as well as $2 \leq p < \infty$, and $0 < \varepsilon < 1/p$ be given fixed. Further let $\Omega \subseteq \mathbb{R}^d$ be either $\mathbb{R}^d$ itself, a bounded Lipschitz domain, or an interval (if $d = 1$). Since Ω is open it contains inner points. By a simple translation and dilation argument (see, e.g. [9, Sect. 4]) we may w.l.o.g. assume that the Euclidean ball of radius one, $B_1(0)$, is contained in Ω. In what follows we will choose specific values $\sigma \in (0, 1)$ as well as $\theta \in (1, \infty)$ and define $u := u_{\sigma,\theta,d}$ and $f := f^{[p]}_{\sigma,\theta,d}$ according to (16). From Lemma 2.6 it then follows that $u \in W^1_{p,0}(\Omega)$ is a weak solution to (1) with right-hand side $f \in W^{-1}_{p'}(\Omega)$ and that the supports of u and f are contained in $B_{4/5}(0)$.

Given $1 < \mu \leq \infty$ and $\varepsilon\,(p-1) < \lambda < 1 - \varepsilon$ we choose θ such that $0 < 1 - \varepsilon < 1/\theta < 1$ and define

$$\sigma := \frac{\lambda}{p-1} - \frac{1 - 1/\theta}{(p-1)\,\mu}.$$

Then it is easily seen that

$$0 < \frac{\lambda}{p-1} - \varepsilon < \sigma \leq (p-1)\,\sigma < \frac{1}{\theta} < 1. \qquad (25)$$

Indeed, the lower bound on λ shows that $\lambda/(p-1)-\varepsilon$ is strictly positive. If $1<\mu<\infty$, we note that $p\geq 2$ implies $1-\varepsilon\,(p-1)\,\mu<1-\varepsilon<1/\theta$ and hence

$$\frac{\lambda}{p-1}-\varepsilon=\frac{\lambda}{p-1}+\frac{1-\varepsilon\,(p-1)\,\mu-1}{(p-1)\,\mu}<\frac{\lambda}{p-1}+\frac{1/\theta-1}{(p-1)\,\mu}=\sigma,$$

while the corresponding estimate for $\mu=\infty$ is trivial since $\varepsilon>0$. Moreover, $1/\theta<1$ yields

$$\sigma\leq(p-1)\,\sigma\leq\lambda<1-\varepsilon<\frac{1}{\theta}<1$$

which completes the proof of (25).

Next we note that (25) particularly implies that

$$\min\left\{\theta\,(1+\sigma),\frac{1-\sigma}{1-1/\theta}\right\}>1.$$

So, we can employ Lemma 2.3(iii) to see that for $1\leq\varrho\leq\infty$ there holds

$$w_{\sigma,\theta}\in B^{s}_{\varrho,q}(\mathbb{R})\quad\text{if and only if}\quad s=s_\varrho\ \text{ and }\ q=\infty,\quad\text{or}\quad s<s_\varrho\ \text{ and }\ 0<q\leq\infty,\tag{26}$$

where

$$s_\varrho:=\sigma+\frac{1-1/\theta}{\varrho}=\frac{\lambda}{p-1}+\left(\frac{1}{\varrho}-\frac{1}{(p-1)\,\mu}\right)\left(1-\frac{1}{\theta}\right).$$

Note that our assumptions imply that $0<1-1/\theta<\varepsilon$,

$$\frac{1}{\varrho}-\frac{1}{(p-1)\,\mu}\in\begin{cases}(-1,0) & \text{if}\quad (p-1)\,\mu<\varrho\leq\infty,\\ \{0\} & \text{if}\quad \varrho=(p-1)\,\mu,\\ (0,1) & \text{if}\quad 1<\varrho<(p-1)\,\mu,\end{cases}$$

as well as $0<\lambda/(p-1)\pm\varepsilon<1$. If ϱ is large, we have $\lambda/(p-1)-\varepsilon<s_\varrho<\lambda/(p-1)$. Thus (26) and Lemma 2.8(i) prove

$$u=u_{\sigma,\theta,d}\in B^{1+\frac{\lambda}{p-1}-\varepsilon}_{\varrho,q}(\Omega)\setminus B^{1+\frac{\lambda}{p-1}}_{\varrho,q}(\Omega),\qquad 0<q\leq\infty,$$

for this case. The regularity statements for u in the remaining cases are obtained likewise.

Similarly, (25) and Lemma 2.3(iii) show that for $1\leq\varrho\leq\infty$ there holds

$$w_{(p-1)\sigma,\theta}\in B^{s}_{\varrho,q}(\mathbb{R})\quad\text{if and only if}\quad s=\widetilde{s}_\varrho\ \text{ and }\ q=\infty,\quad\text{or}\quad s<\widetilde{s}_\varrho\ \text{ and }\ 0<q\leq\infty,$$

where now (depending on the relation of ϱ and μ to each other)

$$\widetilde{s}_\varrho:=(p-1)\,\sigma+\frac{1-1/\theta}{\varrho}=\lambda+\left(\frac{1}{\varrho}-\frac{1}{\mu}\right)\left(1-\frac{1}{\theta}\right)\in(\lambda-\varepsilon,\lambda+\varepsilon)\subsetneq(0,1).$$

Therefore, we can use Lemma 2.8(ii) to deduce the regularity statements for $A(\nabla u)$. In particular we have $A(\nabla u)\in\left(B^{\lambda}_{\mu,\infty}(\Omega)\right)^d$ such that, by Lemma 2.8(iii), $f=f^{[p]}_{\sigma,\theta,d}\in B^{-1+\lambda}_{\mu,\infty}(\Omega)$, as claimed. ■

2.3 Proof of Theorem 1.6

In order to show Theorem 1.6, we essentially follow the lines of the proof of Theorem 1.3. So let us focus on the necessary modifications only.

Proof. Given $1 < \mu \leq \infty$ and $0 < \varepsilon < \min\{1/(p-1), 1-1/(p-1)\}$ (note that this time $p > 2!$) we choose θ such that $0 < 1-\varepsilon < 1/\theta < 1$ and define

$$\sigma := \frac{1}{p-1} - \frac{1-1/\theta}{(p-1)\,\mu}.$$

Then there holds

$$0 < \sigma < \frac{1}{\theta} < (p-1)\,\sigma \leq 1.$$

Indeed, $\mu > 1$ and $0 < 1/\theta < 1$ show that $0 \leq (1-1/\theta)/\mu < 1-1/\theta < 1$. This proves $0 < \sigma$, as well as $1/\theta < (p-1)\,\sigma \leq 1$. Hence, we also have $\sigma \leq 1/(p-1) < 1-\varepsilon < 1/\theta$ due to our assumption on ε.

Now the claimed regularity of u follows exactly as in the proof of Theorem 1.3, where this time our restrictions on ε ensure that $0 < 1/(p-1) \pm \varepsilon < 1$.

In order to prove the regularity statement for $A(\nabla u)$ we like to apply Lemma 2.8(ii). To this end, we have to show that $w_{(p-1)\sigma,\theta} \in W^1_\varrho(\mathbb{R})$ for $1 < \varrho < \infty$ if and only if $\varrho < \mu$. By Lemma 2.3(i) this reduces to the claim $w'_{(p-1)\sigma,\theta} \in L_\varrho(\mathbb{R})$. If $\mu = \infty$, then we actually have $(p-1)\,\sigma = 1$. Therefore, from Lemma 2.3(ii) it follows that $w'_{(p-1)\sigma,\theta} \in L_\varrho(\mathbb{R})$ for all $0 < \varrho \leq \infty$. On the other hand, if $\mu < \infty$, then $0 < (p-1)\,\sigma < 1$. Hence, in this case we have $w'_{(p-1)\sigma,\theta} \in L_\varrho(\mathbb{R})$ if and only if

$$\frac{1-(p-1)\,\sigma}{1-1/\theta} = \frac{1}{\mu} < \frac{1}{\varrho}.$$

In conclusion, $A(\nabla u) \in \left(W^1_\varrho(\Omega)\right)^d$ for $1 < \varrho < \infty$ is equivalent to $\varrho < \mu$, as claimed.

It remains to prove that $f = f^{[p]}_{\sigma,\theta,d}$ belongs to $L_\nu(\Omega)$. If $\nu > 1$, this follows from Lemma 2.8(iv) and the calculations above. However, in view of the compact support of f, this lower bound on ν can be dropped. ∎

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