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# HIGHER ORDER CALDERÓN-ZYGMUND ESTIMATES FOR THE $p$ -LAPLACE EQUATION

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ABSTRACT. The paper is concerned with higher order Calderón-Zygmund estimates for the  $p$ -Laplace equation

$$-\operatorname{div}(A(\nabla u)) := -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = -\operatorname{div} F, \quad 1 < p < \infty.$$

We are able to transfer local interior Besov and Triebel-Lizorkin regularity up to first order derivatives from the force term  $F$  to the flux  $A(\nabla u)$ . For  $p \geq 2$  we show that  $F \in \mathbf{B}_{\varrho,q}^s$  implies  $A(\nabla u) \in \mathbf{B}_{\varrho,q}^s$  for any  $s \in (0, 1)$  and all reasonable  $\varrho, q \in (0, \infty]$  in the planar case. The result fails for  $p < 2$ . In case of higher dimensions and systems we have a smallness restriction on  $s$ . The quasi-Banach case  $0 < \min\{\varrho, q\} < 1$  is included, since it has important applications in the adaptive finite element analysis. As an intermediate step we prove new linear decay estimates for  $p$ -harmonic functions in the plane for the full range  $1 < p < \infty$ .

## 1. INTRODUCTION

In this paper we study Calderón-Zygmund type estimates for the weak solution of the  $p$ -Poisson equation

$$(1.1) \quad -\operatorname{div}(A(\nabla u)) := -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = -\operatorname{div} F \quad \text{in } \Omega,$$

where  $d, n \in \mathbb{N}$ ,  $\Omega$  is an open set in  $\mathbb{R}^d$ ,  $1 < p < \infty$ , and  $u: \Omega \rightarrow \mathbb{R}^n$  is the unknown. All our results are of local nature so no boundary conditions are required. Most of them are restricted to  $p \geq 2$  and scalar solutions ( $n = 1$ ) for  $d = 2$ .

The main objective in non-linear Calderón-Zygmund theory is to transfer the regularity of the right hand side  $F$  to the flux  $A(\nabla u)$  (or to  $\nabla u$  itself) in the norm of an appropriate function space  $\mathbf{X}$ . The corresponding estimate can be written as

$$(1.2) \quad \|A(\nabla u)\|_{\mathbf{X}} \leq C \|F\|_{\mathbf{X}},$$

or, in its local version,

$$(1.3) \quad \|A(\nabla u)\|_{\mathbf{X}(B)} \leq C \|F\|_{\mathbf{X}(2B)} + \text{lower order terms of } A(\nabla u),$$

where  $B$  denotes an arbitrary ball such that  $2B \subset \Omega$ .

The choice  $\mathbf{X} = L^{p'}$  (with  $\frac{1}{p} + \frac{1}{p'} = 1$ ) corresponds to the standard estimates of weak solutions. The first breakthrough was the result of [23], who showed the estimate (1.2) for  $\mathbf{X} = L^r$  and all  $r \in [p', \infty)$ . Later on this result was extended to  $\mathbf{X} = \text{BMO}$  for  $p \geq 2$  in [12] and for an arbitrary exponent  $p > 1$  in [15]. It became clear from the calculations in [15] that it is better to look at the mapping  $F \mapsto A(\nabla u)$  rather than  $F \mapsto \nabla u$ . This is also supported by [24], where potential estimates for the mapping  $f := \operatorname{div} F \mapsto A(\nabla u)$  have been studied. Moreover, it has been shown in [15] that it is possible to take  $\mathbf{X} = \text{BMO}_\omega$ , or  $\mathbf{X} = C^{0,\alpha}$ , resp., as long as the modulus of continuity  $\omega$ , resp.  $\alpha > 0$ , satisfies some smallness condition which depends on the

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best known regularity of  $p$ -harmonic functions. In particular, for  $d \geq 3$  or vectorial solutions the exponent  $\alpha > 0$  is just an unknown small quantity.

In this paper we extend the Calderón-Zygmund estimates for  $p \geq 2$  and  $d = 2$  to spaces of differentiability up to order one. In particular, we show that the estimate (1.3) holds true also for Besov spaces  $\mathbf{X} = \mathbf{B}_{\varrho,q}^s$  for all exponents of smoothness  $s \in (0, 1)$ , every integrability parameter  $\varrho \in (0, \infty]$ , and all fine indices  $q \in (0, \infty]$  such that  $\mathbf{B}_{\varrho,q}^s \hookrightarrow L^{p'}$ . Moreover, if additionally  $\varrho < \infty$ , then a similar assertion remains valid in the scale of Triebel-Lizorkin spaces  $\mathbf{X} = \mathbf{F}_{\varrho,q}^s$ . We refer to Theorem 4.1 for the precise statements. Let us stress the fact that these scales include a lot of classical function spaces such as, e.g., Hölder-Zygmund, Bessel-potential, or Sobolev-Slobodeckij spaces, as special cases [25]. The restriction  $p \geq 2$  in our result is natural in this context, since the assertion fails for  $1 < p < 2$  even for  $F \equiv 0$  and  $d = 2$ , see Subsection 2.6. The assumed compact embedding in  $L^{p'}$  ensures that we are in the context of weak solutions, i.e., that  $u \in W_{\text{loc}}^{1,p}(\Omega)$ . In the case  $d \geq 3$  we obtain similar results, but then there are restrictions on  $s$  due to some open problems (see Subsection 2.7) on the regularity of  $p$ -harmonic functions in higher dimensions.

Our work is motivated by the numerical analysis of the  $p$ -Poisson equation using wavelets or the adaptive finite element method. Note that the approximability of the solutions by discrete ones is determined solely by the differentiability  $s$  from  $\mathbf{B}_{\varrho,q}^s$ . We refer to [9, 11] for a detailed study of numerical approximability. However, in many cases it is possible to increase  $s$  by decreasing the integrability  $\varrho$ , where the strongest results are obtained if we take  $\varrho < 1$ . Then we are in the regime of quasi-Banach spaces, but nevertheless also in this case the smoothness  $s$  still determines the rates of convergence of best  $N$ -term approximations. For this reason it is important that our estimates cover the full range of parameters  $\varrho, q \in (0, \infty]$ .

Other authors also investigated estimates for  $A(\nabla u)$  in terms of Sobolev or Besov spaces. For example, Cianchi and Maz'ya have shown in [7] that  $F \in W^{1,2}$ , so  $f := \text{div } F \in L^2$ , implies that  $A(\nabla u) \in W^{1,2}$  for any  $d \geq 2$  and any  $1 < p < \infty$ . They also obtain global results under minimal conditions on  $\partial\Omega$ . Moreover, it has been shown by Avelin, Kuusi, and Mingione [3] that  $f \in L^1$  implies that locally  $A(\nabla u) \in W^{s,1}$  for any  $s \in (0, 1)$ ,  $d \geq 2$  and  $p > 2 - \frac{1}{d}$ . For  $p \leq d$  this requires the concept of so-called *solutions obtained as limits of approximations* (SOLA). Both results support the fact that the mappings  $F \mapsto A(\nabla u)$  and  $f \mapsto A(\nabla u)$  are the natural ones. Our regularity results differ from [7] and [3] in the sense that we provide estimates for all integrability exponents  $\varrho$  (from  $\mathbf{B}_{\varrho,q}^s$  or  $\mathbf{F}_{\varrho,q}^s$ ), while [7] is restricted to  $\varrho = 2$  and [3] is restricted to  $\varrho = 1$ . Let us mention again that estimates for arbitrary exponents  $\varrho$  are only possible for  $p \geq 2$ , see Subsection 2.6.

In Subsection 4.2 we discuss how our results on the regularity of  $A(\nabla u)$  translate into regularity assertions for  $\nabla u$  and  $V(\nabla u) = |\nabla u|^{\frac{p-2}{2}} \nabla u$ . This allows us also to compare our results with the work of other authors on the higher differentiability of these quantities. For example, it has been shown in [10] that for  $d = 2$  and  $f \in L^\infty$  there holds  $u \in \mathbf{B}_{\varrho,\varrho}^s$  for all  $s \in (0, 2)$  if  $p \leq 2$  and all  $s \in (0, p')$  if  $p > 2$ , where in both cases  $\varrho = \varrho(d, s, p) > 0$  refers to the adaptivity scale of  $L^p$ , i.e.,  $s - \frac{d}{\varrho} = -\frac{d}{p}$ . These results also hold globally on Lipschitz domains with zero boundary data. Corner regularity results with strong conditions on the right-hand side for  $d = 2$  have been studied in [22]. The  $C^{0,\alpha}$ -regularity of  $A(\nabla u)$  up to the boundary for smooth domains has been studied in [5], however it rules out the case of polygonal domains that appear in the context of finite elements. Moreover, it has been shown in [8] that for  $p \geq 2$ ,  $d \geq 2$  and  $s \in (0, 1)$  a forcing term  $F \in \mathbf{B}_{2,q}^s$  implies that  $V(\nabla u) \in \mathbf{B}_{2, \frac{2q}{p}}^{\frac{p's}{2}}$  for  $1 \leq q \leq \frac{2d}{d-2s}$ . For a more detailed comparison of our results to that from [10] and [8] we refer to Subsection 4.2.

The main idea of our proof is to employ a well-known characterization of Besov and Triebel-Lizorkin spaces in terms of oscillations, see Lemma 4.3. This allows to reduce the proof of (1.3) to an oscillation decay estimate for  $A(\nabla u)$ . This fundamental decay estimate is formulated in Theorem 3.1. It is of independent interest since it allows to significantly improve the decay estimate from [6] at least in the case of the plane.

Certainly, the oscillation estimates for  $A(\nabla u)$  can never be better than the ones for  $p$ -harmonic functions, i.e., for  $h$  with  $\operatorname{div}(A(\nabla h)) = 0$ , which corresponds to the case  $F \equiv 0$ . In two dimensions we are able to prove a new (almost) linear decay estimate for the oscillations of  $A(\nabla h)$  for  $p \geq 2$ . Indeed, in Theorem 2.2 we show that

$$\int_{\theta B} |A(\nabla h) - \langle A(\nabla h) \rangle_{\theta B}| dx \leq c_\beta \theta^\beta \int_B |A(\nabla h) - \langle A(\nabla h) \rangle_B| dx$$

for any  $\theta, \beta \in (0, 1)$ . From this we deduce by duality decay estimates for  $\nabla h$  in the case  $1 < p \leq 2$ , see Theorem 2.3. It has been shown by Iwaniec and Manfredi in [23] that  $A(\nabla h) \in C^1$  for  $p \geq 2$  while  $\nabla h \in C^1$  if  $p \leq 2$ . However, the techniques therein do not provide qualitative decay estimates. Instead, we use and improve the approach of [1] and [4], which allows us to obtain new decay estimates. We also implement several ideas from [6].

The paper is organized as follows: In Section 2 we study the regularity of  $p$ -harmonic functions in the plane. Here we deduce the important decay estimates for  $A(\nabla h)$  that we shall need later. Starting from Section 3 we study the  $p$ -Poisson equation with a force term  $\operatorname{div} F$ . We derive in this section the crucial oscillation estimates of  $A(\nabla u)$  in terms of the oscillations of  $F$ . In Section 4 we prove the nonlinear Calderón-Zygmund estimates that allow to transfer  $\mathbf{B}_{\varrho, q}^s$ , resp.  $\mathbf{F}_{\varrho, q}^s$  regularity from  $F$  to  $A(\nabla u)$ . Here we also explain how the regularity of  $A(\nabla u)$  implies regularity of  $\nabla u$  and  $V(\nabla u)$ . Throughout the paper we assume  $p \geq 2$ . Only in Subsection 2.6 we deal with the case  $1 < p < 2$  and present a new decay estimate.

## 2. REGULARITY OF $p$ -HARMONIC FUNCTIONS

Regularity studies of solution to the problem (2.1) are about 50 years old. They go back to Ural'tseva [27], where it was shown that  $p$ -harmonic functions belong to the local Hölder class  $C_{\text{loc}}^{1, \alpha}(\Omega)$  for some exponent  $\alpha = \alpha(d, p) < 1$ . For the case  $d = 2$  the sharp value of the Hölder exponent  $\alpha$  is known, see [23], while for  $d \geq 3$  this problem is still open.

Before we proceed let us first introduce some notation. For vectors  $Q$  we define  $A$  and  $V$  in the following way:

$$\begin{aligned} A(Q) &= |Q|^{p-2} Q, \\ V(Q) &= |Q|^{\frac{p-2}{2}} Q, \end{aligned}$$

where  $|\cdot|$  denotes the Euclidean norm. Note that  $A$  and  $V$  are isomorphisms. Moreover, by  $B$ ,  $B_r$ , and  $B_r(x)$  we usually denote open Euclidean balls with radius  $r > 0$  and center  $x \in \Omega \subset \mathbb{R}^d$ . We write  $\lambda B$  for the ball with same center as  $B$  but scaled in size by  $\lambda$ . Further, for  $f \in L_{\text{loc}}^1(\mathbb{R}^d)$  we define the mean value over the ball  $B$  as

$$\langle f \rangle_B := \int_B f dx,$$

where  $\int_B \cdots dx := |B|^{-1} \int_B \cdots dx$  denotes the average integral with  $|B|$  being the volume of  $B$ . The same notation is employed also in the vector-valued case. Moreover, we shall use  $c$  as a generic positive constant which may change from line to line, but does not depend on the crucial quantities. We will use the notation  $f \lesssim g$  if there exist a constant such that  $f \leq cg$ . Finally, we write  $f \approx g$  if  $f \lesssim g$  and  $g \lesssim f$ .

**Definition 2.1.** *A function  $h: \Omega \rightarrow \mathbb{R}$  is called  $p$ -harmonic in  $\Omega \subset \mathbb{R}^d$  if it is a weak solution of the  $p$ -harmonic equation, i.e.,  $h \in W_{\text{loc}}^{1, p}(\Omega)$  and*

$$(2.1) \quad -\operatorname{div}(A(\nabla h)) = 0$$

*in the distributional sense.*

Throughout the paper we will use the letters  $h$  for  $p$ -harmonic functions and  $u$  for solutions to the  $p$ -Poisson equation (1.1).

The main result of this section is the following decay estimate for  $A(\nabla h)$ .

**Theorem 2.2.** *Let  $h: \Omega \rightarrow \mathbb{R}$  be  $p$ -harmonic with  $p \geq 2$  on  $\Omega \subset \mathbb{R}^2$ . Then for all  $\beta \in (0, 1)$ , there exists  $c_\beta > 0$  such that for all balls  $B \subset \Omega$  and all  $\theta \in (0, 1)$  there holds*

$$\int_{\theta B} |A(\nabla h) - \langle A(\nabla h) \rangle_{\theta B}| dx \leq c_\beta \theta^\beta \int_B |A(\nabla h) - \langle A(\nabla h) \rangle_B| dx.$$

In Proposition 2.15 below we present a corresponding estimate with power  $p'$  on the left-hand side.

The proof of Theorem 2.2 requires a few preliminary steps. The basic idea is to distinguish between the *non-degenerate* and the *degenerate* case. The non-degenerate case is the one where

$$\int_B |V(\nabla h) - \langle V(\nabla h) \rangle_B|^2 dx \leq \varepsilon_{\text{DG}} \int_B |V(\nabla h)|^2 dx$$

for a suitable small  $\varepsilon_{\text{DG}} > 0$ . In particular,  $V(\nabla h)$  is (in average) close to the constant  $\langle V(\nabla h) \rangle_B$ , so  $\nabla h$  is also close to a constant. In this case  $A(\nabla h) \approx \langle \nabla h \rangle_B^{p-2} \nabla h$ , so the equation behaves locally like a linear equation with constant coefficients and we get our decay estimates from this. See Subsection 2.3 for details. In contrast, for the degenerate case we have to argue differently. In this situation we will use certain decay estimates of quasi-conformal gradient maps which also explain the restriction to  $d = 2$ , see Subsection 2.4.

Most of our results are restricted to the case  $p \geq 2$ . However, the following remarkable decay estimate for the case  $1 < p \leq 2$  is obtained in Subsection 2.6 by a duality argument.

**Theorem 2.3.** *Let  $h: \Omega \rightarrow \mathbb{R}$  be  $p$ -harmonic with  $1 < p \leq 2$  in  $\Omega \subset \mathbb{R}^2$ . Then for all  $\beta \in (0, 1)$ , there exists  $c_\beta > 0$  such that for all balls  $B \subset \Omega$  and all  $\theta \in (0, 1)$  there holds*

$$\int_{\theta B} |\nabla h - \langle \nabla h \rangle_{\theta B}| dx \leq c_\beta \theta^\beta \int_B |\nabla h - \langle \nabla h \rangle_B| dx.$$

**Remark 2.4.** *The Theorems 2.2 and 2.3 improve the decay results from [15, Remark 5.6] significantly in the situation of the plane. Indeed, the result in [15] is restricted to  $\beta \in (0, \beta_0)$ , where  $\beta_0 > 0$  is some unknown small number.*

**2.1. Shifted Orlicz functions and monotonicity.** In this subsection we introduce shifted N-functions and present some monotonicity estimates.

For  $1 < p < \infty$  we define  $\varphi: [0, \infty) \rightarrow [0, \infty)$  by

$$\varphi(t) := \frac{1}{p} t^p.$$

Then  $\varphi$  is a so-called N-function, i.e. there exists a derivative  $\varphi'$  of  $\varphi$  which is right continuous, non-decreasing, and satisfies  $\varphi'(0) = 0$ , as well as  $\varphi'(t) > 0$  for  $t > 0$ . In particular,  $\varphi$  is convex. By  $\varphi^*$  we denote the complementary N-function, i.e.,

$$\varphi^*(t) := \sup_{s \geq 0} (st - \varphi(s)).$$

Especially, in our setting, there holds  $\varphi^*(t) = \frac{1}{p'} t^{p'}$  with  $\frac{1}{p} + \frac{1}{p'} = 1$ .

We will use a technique based on the properties of *shifted* N-functions, introduced in [13, 16]:

**Definition 2.5.** *For  $a, t \geq 0$  we define the shifted N-functions  $\varphi_a$  as*

$$\varphi_a(t) := \int_0^t \frac{\varphi'(\max\{a, s\})}{\max\{a, s\}} ds.$$

**Remark 2.6.** *We use here the version of [14] that is equivalent to the original one, where  $\max\{a, s\}$  is replaced by  $a + s$ . This new version however has a few simple advantages, e.g.,  $(\varphi_a)^* = \varphi_{\varphi'(a)}^*$  instead of  $(\varphi_a)^* \approx \varphi_{\varphi'(a)}^*$ . Note that in our notation the  $*$  binds stronger than the shift index. That is, we let  $\varphi_a^* := (\varphi^*)_a$ .*

Choosing  $\varphi$  as above, we have the following equivalences for  $a, t \geq 0$

$$\begin{aligned} \varphi_a(t) &\approx (a+t)^{p-2} t^2, \\ \varphi'_a(t) &\approx (a+t)^{p-2} t, \\ (\varphi_a)^*(t) &= \varphi_{\varphi'(a)}^*(t) \approx (a^{p-1} + t)^{p'-2} t^2. \end{aligned}$$

It is important to observe, that the family  $\{\varphi_a\}_a$  satisfies a uniform  $\Delta_2$  and  $\nabla_2$ -condition, i.e., uniformly in  $a, t \geq 0$  there holds  $\varphi_a(2t) \approx \varphi_a(t)$  and  $\varphi_a^*(2t) \approx \varphi_a^*(t)$ . This implies that Young's inequality holds independently of the shift, i.e. for every  $\delta > 0$  there exists  $c_\delta$  such that for all  $s, t, a \geq 0$  there holds

$$\varphi'_a(s) t \leq \delta \varphi_a(s) + c_\delta \varphi_a(t).$$

In the following auxiliary statements we recall some well-known connections between  $A$ ,  $V$ , and shifted  $N$ -functions. For the proofs we refer to [13, Lemma 3], [16, Appendix] and [14]. Here and in what follows  $\cdot$  denotes the Euclidean scalar product.

**Lemma 2.7** (Monotonicity). *For  $1 < p < \infty$  we have*

$$\begin{aligned} (A(P) - A(Q)) \cdot (P - Q) &\approx |V(P) - V(Q)|^2 \approx (|Q| + |P|)^{p-2} |Q - P|^2 \\ &\approx \varphi_{|Q|}(|P - Q|) \approx \varphi_{|A(Q)|}(|A(P) - A(Q)|), \end{aligned}$$

as well as

$$|A(P) - A(Q)| \approx \varphi'_{|Q|}(|P - Q|) \quad \text{and} \quad A(Q) \cdot Q = |V(Q)|^2 \approx \varphi(|Q|).$$

Using the fact that the function  $t \mapsto (|Q| + t)^{p-2}$  is increasing for  $p \geq 2$ , we immediately obtain the following corollary:

**Corollary 2.8.** *Let  $p \geq 2$ , then for all vectors  $P$  and  $Q$  there holds*

$$|P - Q|^{p'} \lesssim (A(P) - A(Q)) \cdot (P - Q) \lesssim |A(P) - A(Q)|^{p'}$$

and for all  $t, a \geq 0$  we have  $\varphi_a^*(t) \leq ct^{p'}$ .

The next lemma is taken from [15, Lemma 2.5]. It is a refined version from the one in [16] and shows how to perform a “shift-change”.

**Lemma 2.9.** *Let  $1 < p < \infty$ . Then for all vectors  $P, Q$  and every  $\lambda \in (0, 1]$  it holds*

$$\begin{aligned} \varphi_{|P|}(t) &\leq c \lambda^{1-\max\{p', 2\}} \varphi_{|Q|}(t) + \lambda |V(P) - V(Q)|^2, \\ \varphi_{|A(P)|}^*(t) &\leq c \lambda^{1-\max\{p, 2\}} \varphi_{|A(Q)|}^*(t) + \lambda |V(P) - V(Q)|^2, \end{aligned}$$

where the constants only depend on  $p$ .

Moreover, we will make frequent use of the following well-known estimate.

**Lemma 2.10.** *Let  $1 \leq q < \infty$ . Then for all balls  $B$  and  $G \in L^q(B)$ , there holds*

$$\inf_{G_0} \left( \int_B |G - G_0|^q dx \right)^{\frac{1}{q}} \leq \left( \int_B |G - \langle G \rangle_B|^q dx \right)^{\frac{1}{q}} \leq 2 \inf_{G_0} \left( \int_B |G - G_0|^q dx \right)^{\frac{1}{q}},$$

where the infima are taken over all constants  $G_0$ . If  $q = 2$ , then we have equality in the first estimate. It is possible to replace  $B$  by an arbitrary set of positive measure.

Finally, given a gradient  $\nabla v$ , we define its  $A$ - and  $V$ -averages  $\langle \nabla v \rangle_B^A$ , resp.  $\langle \nabla v \rangle_B^V$ , on balls  $B$  by the relations

$$A(\langle \nabla v \rangle_B^A) = \langle A(\nabla v) \rangle_B \quad \text{and} \quad V(\langle \nabla v \rangle_B^V) = \langle V(\nabla v) \rangle_B$$

and note that both of them are well-defined, since  $A(\cdot)$  and  $V(\cdot)$  are isomorphisms. Then the next result, taken from [15, Lemma A.2], states that the mean oscillations of  $V(\nabla v)$  with respect to the different versions of averages are equivalent.

**Lemma 2.11.** *Assume that  $\nabla v \in L^p(B)$  for a given ball  $B$ . Then*

$$\int_B |V(\nabla v) - V(\langle \nabla v \rangle_B^V)|^2 \approx \int_B |V(\nabla v) - V(\langle \nabla v \rangle_B)|^2 dx \approx \int_B |V(\nabla v) - V(\langle \nabla v \rangle_B^A)|^2 dx.$$

**2.2. Reverse Hölder's estimate.** In this subsection we show that it is possible to measure the oscillations of  $A(\nabla h)$  with or without the power  $p'$  for  $p \geq 2$ .

Let us begin with the following estimate of reverse Hölder type.

**Lemma 2.12** ([15, Corollary 3.5]). *If  $h$  is  $p$ -harmonic with  $p \geq 2$ , then for all  $Q$*

$$\int_B |V(\nabla h) - V(Q)|^2 dx \lesssim \varphi_{|A(Q)|}^* \left( \int_{2B} |A(\nabla h) - A(Q)| dx \right).$$

By combining this with Lemma 2.7, it follows that

$$\int_B \varphi_{|A(Q)|}^* (|A(\nabla h) - A(Q)|) dx \lesssim \varphi_{|A(Q)|}^* \left( \int_{2B} |A(\nabla h) - A(Q)| dx \right).$$

If we now apply the inverse of  $\varphi_{|A(Q)|}^*$  to both sides, then we obtain

$$(2.2) \quad (\varphi_{|A(Q)|}^*)^{-1} \left( \int_B \varphi_{|A(Q)|}^* (|A(\nabla h) - A(Q)|) dx \right) \lesssim \int_{2B} |A(\nabla h) - A(Q)| dx.$$

In order to proceed further, we shall need the following auxiliary lemma.

**Lemma 2.13.** *If  $p \geq 2$ , then for all  $Q$  we have*

$$\left( \int_B |f|^{p'} dx \right)^{\frac{1}{p'}} \lesssim (\varphi_{|A(Q)|}^*)^{-1} \left( \int_B \varphi_{|A(Q)|}^* (|f|) dx \right).$$

*Proof.* We estimate

$$\psi(t) := \varphi_{|A(Q)|}^* (t^{\frac{1}{p'}}) \approx \left( |A(Q)| + t^{\frac{1}{p'}} \right)^{p'-2} t^{\frac{2}{p'}} \approx \begin{cases} |A(Q)|^{p'-2} t^{\frac{2}{p'}} & \text{for } t^{\frac{1}{p'}} \leq |A(Q)|, \\ t & \text{for } t^{\frac{1}{p'}} > |A(Q)|. \end{cases}$$

Now  $\psi$  is continuous and convex. Thus, by Jensen's inequality,

$$\varphi_{|A(Q)|}^* \left( \left( \int_B |f|^{p'} dx \right)^{\frac{1}{p'}} \right) \approx \psi \left( \int_B |f|^{p'} dx \right) \geq \int_B \psi(|f|^{p'}) dx \approx \int_B \varphi_{|A(Q)|}^* (|f|) dx.$$

This proves the claim.  $\square$

**Proposition 2.14** (Reverse Hölder). *Let  $h$  be  $p$ -harmonic with  $p \geq 2$ . Then*

$$\left( \int_B |A(\nabla h) - \langle A(\nabla h) \rangle_B|^{p'} dx \right)^{\frac{1}{p'}} \lesssim \int_{2B} |A(\nabla h) - \langle A(\nabla h) \rangle_{2B}| dx.$$

*Proof.* We define  $Q$  by  $A(Q) := \langle A(\nabla h) \rangle_{2B}$ . Then the combination of (2.2) and Lemma 2.13 proves the claim with the mean value on the left-hand side replaced by  $\langle A(\nabla h) \rangle_{2B}$ . Due to Lemma 2.10 we can exchange it by  $\langle A(\nabla h) \rangle_B$  which completes the proof.  $\square$

Overall, we see that the oscillation of  $A(\nabla h)$  can be measured with power 1 or  $p'$ . In particular, if we combine Theorem 2.2 (which still has to be proven) and Proposition 2.14, we get the following decay estimate in powers of  $p'$ .

**Proposition 2.15.** *Let  $h: \Omega \rightarrow \mathbb{R}$  be  $p$ -harmonic with  $p \geq 2$  on  $\Omega \subset \mathbb{R}^2$ . Then for all  $\beta \in (0, 1)$  there exists  $c_\beta > 0$  such that for all balls  $B \subset \Omega$  and all  $\theta \in (0, 1)$  there holds*

$$\left( \int_{\theta B} |A(\nabla h) - \langle A(\nabla h) \rangle_{\theta B}|^{p'} dx \right)^{\frac{1}{p'}} \leq c_\beta \theta^\beta \int_B |A(\nabla h) - \langle A(\nabla h) \rangle_B| dx.$$

**2.3. Non-degenerate case.** Let us consider the non-degenerate case. That is, we will assume that for some fixed ball  $B$  there holds

$$(2.3) \quad \int_B |V(\nabla h) - \langle V(\nabla h) \rangle_B|^2 dx \leq \varepsilon_{\text{DG}} \int_B |V(\nabla h)|^2 dx$$

with some suitably small  $\varepsilon_{\text{DG}} > 0$ .

Inequality (2.3) means that in this situation  $V(\nabla h)$  (and therefore also  $\nabla h$ ) behaves almost like a constant on  $B$ . In particular,  $A(\nabla h) \approx |\langle \nabla h \rangle_B|^{p-2} \nabla h$ . Hence, it is possible to treat the  $p$ -Laplace equation like a perturbation of a linear equation with constant coefficients. This approach was used in [17, Proposition 28] to prove (almost) linear decay estimates of the oscillation in this non-degenerate situation. In fact, it was shown that the decay estimate for the oscillations of  $V$  even holds in the case of quasi-convex functionals with Orlicz growth (for any dimension). Our situation is just a special case. In particular, we obtain:

**Lemma 2.16.** *For every  $\beta \in (0, 1)$  there exists  $c = c(p, \beta) > 0$  and  $\varepsilon_{\text{DG}} = \varepsilon_{\text{DG}}(p, \beta) > 0$  such that if the non-degeneracy condition (2.3) holds, then*

$$\left( \int_{\theta B} |V(\nabla h) - \langle V(\nabla h) \rangle_{\theta B}|^2 dx \right)^{\frac{1}{2}} \leq c \theta^\beta \left( \int_B |V(\nabla h) - \langle V(\nabla h) \rangle_B|^2 dx \right)^{\frac{1}{2}}$$

for all  $\theta \in (0, 1]$ .

From this (almost) linear decay of the oscillations of  $V$  in the non-degenerate case, we will now derive (almost) linear decay of the oscillations of  $A$ .

**Proposition 2.17.** *For every  $\beta \in (0, 1)$ , there exists  $c = c(p, \beta) > 0$  and  $\varepsilon_{\text{DG}} = \varepsilon_{\text{DG}}(p, \beta) > 0$  such that if the non-degeneracy condition (2.3) holds, then*

$$\int_{\theta B} |A(\nabla h) - \langle A(\nabla h) \rangle_{\theta B}| dx \leq c \theta^\beta \int_B |A(\nabla h) - \langle A(\nabla h) \rangle_B| dx$$

for all  $\theta \in (0, 1]$ .

*Proof.* It suffices to prove the claim for  $\theta = 2^{-m}$  with  $m \in \mathbb{N}_0$ . For this purpose, let us define  $B_m := 2^{-m} B$ . Using Lemma 2.11 and Lemma 2.16 we can estimate

$$\begin{aligned} I_m &:= \int_{B_m} |V(\nabla h) - V(\langle \nabla h \rangle_{B_m}^A)|^2 dx \\ &\leq c \int_{B_m} |V(\nabla h) - \langle V(\nabla h) \rangle_{B_m}|^2 dx \\ &\leq c 2^{-2m\beta} \int_{B_1} |V(\nabla h) - \langle V(\nabla h) \rangle_{B_1}|^2 dx \\ &\leq c 2^{-2m\beta} \int_{B_1} |V(\nabla h) - V(\langle \nabla h \rangle_{B_0}^A)|^2 dx. \end{aligned}$$

The reverse Hölder type estimate in Lemma 2.12 with  $Q = \langle \nabla h \rangle_{B_0}^A$  then implies

$$I_m \leq c 2^{-2m\beta} \varphi_{|\langle A(\nabla h) \rangle_{B_0}|}^* \left( \int_B |A(\nabla h) - \langle A(\nabla h) \rangle_{B_0}| dx \right).$$



Using the shift-change Lemma 2.9 (with  $\lambda = 1$ ) and Lemma 2.7 we get

$$\begin{aligned}
I_m &\leq c2^{-2m\beta} \left( \varphi_{|\langle A(\nabla h) \rangle_{B_m}|}^* \left( \int_B |A(\nabla h) - \langle A(\nabla h) \rangle_{B_0}| dx \right) + |V(\langle \nabla h \rangle_{B_0}^A) - V(\langle \nabla h \rangle_{B_m}^A)|^2 \right) \\
&\leq c2^{-2m\beta} \left( \varphi_{|\langle A(\nabla h) \rangle_{B_m}|}^* \left( \int_B |A(\nabla h) - \langle A(\nabla h) \rangle_{B_0}| dx \right) \right. \\
&\quad \left. + \varphi_{|\langle A(\nabla h) \rangle_{B_m}|}^* (|\langle A(\nabla h) \rangle_{B_0} - \langle A(\nabla h) \rangle_{B_m}^A|) \right) \\
&\leq c2^{-2m\beta} \varphi_{|\langle A(\nabla h) \rangle_{B_m}|}^* \left( \int_B |A(\nabla h) - \langle A(\nabla h) \rangle_{B_0}| dx + \theta t |\langle A(\nabla h) \rangle_{B_0} - \langle A(\nabla h) \rangle_{B_m}| \right).
\end{aligned}$$

Since  $p \geq 2$ , we have  $\varphi_a^*(\theta t) \geq c\theta^2 \varphi_a^*(t)$  for all  $a, t \geq 0$  and all  $\theta \in [0, 1]$ . Thus,

$$\begin{aligned}
I_m &\leq c\varphi_{|\langle A(\nabla h) \rangle_{B_m}|}^* \left( 2^{-m\beta} \left( \int_B |A(\nabla h) - \langle A(\nabla h) \rangle_{B_0}| dx \right. \right. \\
(2.4) \quad &\quad \left. \left. + |\langle A(\nabla h) \rangle_{B_0} - \langle A(\nabla h) \rangle_{B_m}| \right) \right).
\end{aligned}$$

On the other hand, with Lemma 2.7 and Jensen's inequality,

$$\begin{aligned}
I_m &= \int_{B_m} |V(\nabla h) - V(\langle \nabla h \rangle_{B_m}^A)|^2 dx \\
&\geq c \int_{B_m} \varphi_{|\langle A(\nabla h) \rangle_{B_m}|}^* (|A(\nabla h) - \langle A(\nabla h) \rangle_{B_m}|) dx \\
&\geq c\varphi_{|\langle A(\nabla h) \rangle_{B_m}|}^* \left( \int_{B_m} |A(\nabla h) - \langle A(\nabla h) \rangle_{B_m}| dx \right)
\end{aligned}$$

We combine this with (2.4) and apply the inverse of  $\varphi_{|\langle A(\nabla h) \rangle_{B_m}|}^*$  to obtain

$$\begin{aligned}
&\int_{B_m} |A(\nabla h) - \langle A(\nabla h) \rangle_{B_m}| dx \\
&\lesssim 2^{-m\beta} \left( \int_{B_0} |A(\nabla h) - \langle A(\nabla h) \rangle_{B_0}| dx + |\langle A(\nabla h) \rangle_{B_0} - \langle A(\nabla h) \rangle_{B_m}| \right).
\end{aligned}$$

Now,

$$\begin{aligned}
|\langle A(\nabla h) \rangle_{B_0} - \langle A(\nabla h) \rangle_{B_m}| &\leq \sum_{k=0}^{m-1} |\langle A(\nabla h) \rangle_{B_k} - \langle A(\nabla h) \rangle_{B_{k+1}}| \\
&\lesssim \sum_{k=0}^{m-1} \int_{B_k} |A(\nabla h) - \langle A(\nabla h) \rangle_{B_k}| dx
\end{aligned}$$

such that from the previous estimate it follows that

$$\int_{B_m} |A(\nabla h) - \langle A(\nabla h) \rangle_{B_m}| dx \leq c2^{-m\beta} \sum_{k=0}^{m-1} \int_{B_k} |A(\nabla h) - \langle A(\nabla h) \rangle_{B_k}| dx, \quad m \in \mathbb{N}.$$

Finally, by Lemma 2.18 below we conclude

$$\int_{B_m} |A(\nabla h) - \langle A(\nabla h) \rangle_{B_m}| dx \leq c_\beta 2^{-m\beta} \int_B |A(\nabla h) - \langle A(\nabla h) \rangle_B| dx$$

and the proof is complete.  $\square$

In the proof of Proposition 2.17 we have used the following algebraic lemma which is shown here for the sake of completeness.

**Lemma 2.18.** *Assume that for some  $c_0, \beta > 0$  the non-negative sequence  $(a_m)_{m \in \mathbb{N}_0}$  satisfies*

$$a_m \leq c_0 2^{-m\beta} \sum_{k=0}^{m-1} a_k, \quad m \in \mathbb{N}.$$

Then for all  $m \in \mathbb{N}$  there holds

$$a_m \leq 2^{-m\beta} \exp\left(\frac{c_0}{1-2^{-\beta}}\right) a_0.$$

*Proof.* Let us define  $b_0 := a_0$  and  $b_m := c_0 2^{-m\beta} \sum_{k=0}^{m-1} b_k$  for  $m \in \mathbb{N}$ . Then by induction there holds  $a_m \leq b_m$ . Moreover, it is  $b_{m+1} - 2^{-\beta} b_m = c_0 2^{-(m+1)\beta} b_m$  such that  $b_{m+1} = 2^{-\beta}(1+c_0 2^{-m\beta})b_m$  and hence

$$b_m = b_0 2^{-m\beta} \prod_{k=0}^{m-1} (1 + c_0 2^{-k\beta}), \quad m \in \mathbb{N},$$

where

$$\prod_{k=0}^{m-1} (1 + c_0 2^{-k\beta}) \leq \exp\left(\sum_{k=0}^{\infty} c_0 2^{-k\beta}\right) = \exp\left(\frac{c_0}{1-2^{-\beta}}\right) =: B.$$

Thus, we have

$$a_m \leq b_m \leq b_0 2^{-m\beta} B = 2^{-m\beta} \exp\left(\frac{c_0}{1-2^{-\beta}}\right) a_0, \quad m \in \mathbb{N},$$

as claimed.  $\square$

**2.4. Degenerate case.** Let us now turn to the degenerate case. We need the following important qualitative regularity result from [1]. Its proof is based on the estimates for quasi-conformal gradient maps from [4].

**Lemma 2.19** ([1]). *Let  $h$  be  $p$ -harmonic with  $p \geq 2$ . Then for all  $\theta \in (0, \frac{1}{2}]$  it holds*

$$\sup_{x, y \in \theta B} |\nabla h(x) - \nabla h(y)| \leq c_0 \theta^\alpha \left( \int_B |\nabla h - \langle \nabla h \rangle_B|^2 dx \right)^{\frac{1}{2}}.$$

where

$$\alpha = \alpha(p) = \frac{1}{2p} \left( -3 - \frac{1}{p-1} + \sqrt{33 + \frac{30}{p-1} + \frac{1}{(p-1)^2}} \right) \geq \frac{1}{p-1}.$$

*Proof.* We will use the following estimate for complex gradients  $\varphi$  from [1, page 546]:

$$[\varphi]_{C^{0,\alpha}(B_{\frac{1}{2}})} \leq c(p, \alpha) 2^{1+\alpha} \|\nabla \varphi\|_{L^2(B_{\frac{1}{2}})},$$

where  $\alpha = \alpha(p)$  and  $c(p, \alpha) = \sqrt{\frac{p-1}{\alpha(p)}}$ . In our notations it follows that

$$\sup_{x, y \in \theta B} |\nabla h(x) - \nabla h(y)| \leq c_0 \theta^\alpha \left( \int_{\frac{3}{4}B} (R |\nabla^2 h|)^2 dx \right)^{\frac{1}{2}},$$

where  $R$  is the radius of  $B$ . Now, the Caccioppoli estimate for quasi-conformal maps proves the claim.  $\square$

**Remark 2.20.** *Note that the exponent  $\alpha = \alpha(p)$  from Lemma 2.19 is smaller than the one in [2, 23]. Unfortunately, these articles do not provide quantitative estimates, so we have to rely on the possibly non-optimal estimate of Lemma 2.19. For example, the regularity for  $p$ -harmonic maps goes to  $C^{0,1/3}$ , as  $p \rightarrow \infty$ , but  $\lim_{p \rightarrow \infty} \alpha(p) = 0$ . Nevertheless, the exponent from Lemma 2.19 is sufficient for (most of) our purposes, since  $\alpha(p) > \frac{1}{p-1}$  for  $p > 2$ . See also the discussions in Subsection 2.7.*

For our purpose we need a  $p$ -version of the previous lemma.

**Lemma 2.21.** *Let  $h$  be  $p$ -harmonic with  $p \geq 2$  let  $\alpha > 0$  be as in Lemma 2.19. Then for all  $\theta \in (0, 1/2]$*

$$\sup_{\theta B} |\nabla h(x) - \nabla h(y)| \leq c_0 \theta^\alpha \left( \int_B |\nabla h - \langle \nabla h \rangle_B|^p dx \right)^{\frac{1}{p}}.$$

*Proof.* This follows from Lemma 2.19 if we apply Jensen's inequality to the right-hand side using  $p \geq 2$ .  $\square$

As a further technical step we also need the following (non-optimal) decay estimate for  $V$  from [18].

**Lemma 2.22** ([18, Theorem 6.4]). *There exists  $\gamma > 0$  such that for all balls  $B$  and all  $\theta \in (0, 1)$  there holds*

$$\left( \int_{\theta B} |V(\nabla h) - \langle V(\nabla h) \rangle_{\theta B}|^2 dx \right)^{\frac{1}{2}} \leq c \theta^\gamma \left( \int_B |V(\nabla h) - \langle V(\nabla h) \rangle_B|^2 dx \right)^{\frac{1}{2}}.$$

**Remark 2.23.** *The decay estimate in Lemma 2.22 is proven for any dimension. However, it provides no explicit lower bound for  $\gamma > 0$ . Therefore, it only provides a very slow, non-optimal decay for  $V$ . See below in Subsection 2.7 for discussions.*

Now we have enough tools at hand to prove an important assertion on alternatives:

**Proposition 2.24.** *Let  $h$  be  $p$ -harmonic with  $p \geq 2$  and let  $\beta \in (0, 1)$ . Suppose that  $h$  fails the non-degeneracy condition (2.3), i.e., we have*

$$\int_B |V(\nabla h) - \langle V(\nabla h) \rangle_B|^2 dx > \varepsilon_{\text{DG}} \int_B |V(\nabla h)|^2 dx.$$

*Then there exist  $\theta_2 = \theta_2(p, \beta, \varepsilon_{\text{DG}}) \in (0, \frac{1}{2})$  and  $m \geq 2$  such that for  $\theta_1 := \theta_2^m$  at least one of the following alternative applies:*

- (a)  *$h$  satisfies the non-degeneracy condition (2.3) on the ball  $\theta_1 B$ .*
- (b)  *$h$  satisfies the decay estimate*

$$\left( \int_{\theta_2 B} |A(\nabla h) - \langle A(\nabla h) \rangle_{\theta_2 B}|^{p'} dx \right)^{\frac{1}{p'}} \leq \theta_2^\beta \left( \int_B |A(\nabla h) - \langle A(\nabla h) \rangle_B|^{p'} dx \right)^{\frac{1}{p'}}.$$

*Proof.* Without loss of generality we can assume that 0 is the center of  $B$ . Suppose that for  $\theta_1$  (to be specified later) alternative (a) fails on  $\theta_1 B$ , i.e., that

$$\int_{\theta_1 B} |V(\nabla h)|^2 dx < \frac{1}{\varepsilon_{\text{DG}}} \int_{\theta_1 B} |V(\nabla h) - \langle V(\nabla h) \rangle_{\theta_1 B}|^2 dx.$$

Then the (non-optimal) decay estimate of  $p$ -harmonic functions of Lemma 2.22 implies that there exists  $\gamma > 0$  such that

$$\int_{\theta_1 B} |V(\nabla h) - \langle V(\nabla h) \rangle_{\theta_1 B}|^2 dx \lesssim \theta_1^{2\gamma} \int_B |V(\nabla h) - \langle V(\nabla h) \rangle_B|^2 dx.$$

So, the  $L^\infty$ -estimate from [18, Lemma 5.8] together with the two previous bounds implies

$$(2.5) \quad |\nabla h(0)|^p \leq c \int_{\theta_1 B} |V(\nabla h)|^2 dx \lesssim \frac{\theta_1^{2\gamma}}{\varepsilon_{\text{DG}}} \int_B |V(\nabla h) - \langle V(\nabla h) \rangle_B|^2 dx$$

Moreover, for the larger ball  $\theta_2 B$  we employ Lemma 2.10 to derive

$$\begin{aligned} \int_{\theta_2 B} |A(\nabla h) - \langle A(\nabla h) \rangle_{\theta_2 B}|^{p'} dx &\lesssim \int_{\theta_2 B} |A(\nabla h)|^{p'} dx \\ &= \int_{\theta_2 B} |\nabla h|^p dx \\ &\lesssim \sup_{x \in \theta_2 B} (|\nabla h(x) - \nabla h(0)|)^p + |\nabla h(0)|^p. \end{aligned}$$

This, Lemma 2.21 (using  $|V(\nabla h)|^2 = |\nabla h|^p$ ), and (2.5) imply that

$$\int_{\theta_1 B} |A(\nabla h) - \langle A(\nabla h) \rangle_{\theta_2 B}|^{p'} dx \lesssim \theta_2^{\alpha p} \int_B |V(\nabla h)|^2 dx + \frac{\theta_1^{2\gamma}}{\varepsilon_{\text{DG}}} \int_B |V(\nabla h) - \langle V(\nabla h) \rangle_B|^2 dx.$$

Since  $h$  fails the non-degeneracy condition (2.3) on  $B$ , we obtain

$$\begin{aligned} \int_{\theta_2 B} |A(\nabla h) - \langle A(\nabla h) \rangle_{\theta_2 B}|^{p'} dx &\lesssim \left( \frac{\theta_2^{\alpha p}}{\varepsilon_{\text{DG}}} + \frac{\theta_1^{2\gamma}}{\varepsilon_{\text{DG}}} \right) \int_B |V(\nabla h) - \langle V(\nabla h) \rangle_B|^2 dx \\ &\leq \frac{c}{\varepsilon_{\text{DG}}} (\theta_2^{\alpha p} + \theta_1^{2\gamma}) \int_B |A(\nabla h) - \langle A(\nabla h) \rangle_B|^{p'} dx, \end{aligned}$$

where for the second estimate we used that  $|V(P) - V(Q)|^2 \lesssim |A(P) - A(Q)|^{p'}$  according to Corollary 2.8 and  $p \geq 2$ .

Let us assume in the following that  $p > 2$ , since the claim of the lemma is standard for the linear case  $p = 2$ . Hence,  $\alpha$  from Lemma 2.21 satisfies  $\alpha > \frac{1}{p-1}$  such that  $\alpha p > p' > \beta p'$ . Therefore we can choose  $\theta_2$  so small that  $c\theta_2^{\alpha p}/\varepsilon_{\text{DG}} \leq \frac{1}{2}\theta_2^{\beta p'}$ . Now choose  $m \geq 2$  large enough such that  $\theta_1 := \theta_2^m$  satisfies  $c\theta_1^{2\gamma}/\varepsilon_{\text{DG}} \leq \frac{1}{2}\theta_2^{\beta p'}$ . So, we finally obtain

$$\int_{\theta_2 B} |A(\nabla h) - \langle A(\nabla h) \rangle_{\theta_2 B}|^{p'} dx \leq \theta_2^{\beta p'} \int_B |A(\nabla h) - \langle A(\nabla h) \rangle_B|^{p'} dx.$$

This proves the claim.  $\square$

**2.5. Proof of Theorem 2.2.** We are now prepared to prove our decay estimates for  $A(\nabla h)$ .

*Proof of Theorem 2.2.* Given  $p \geq 2$  and  $\beta \in (0, 1)$  fix  $\varepsilon_{\text{DG}}$  such that Proposition 2.17 is applicable and choose  $\theta_2 \in (0, \frac{1}{2})$  and  $m \geq 2$  according to Proposition 2.24. Then it is enough to prove the claim for the special sequence of balls  $B_k := \theta_2^k B$ ,  $k \in \mathbb{N}_0$ . It indeed suffices to show that

$$\int_{B_k} |A(\nabla h) - \langle A(\nabla h) \rangle_{B_k}| dx \leq c\theta_2^{k\beta} \int_{B_0} |A(\nabla h) - \langle A(\nabla h) \rangle_{B_0}| dx, \quad k \in \mathbb{N} \setminus \{1\}.$$

To this end, let  $k_0 \in \mathbb{N}_0$  denote the smallest number  $k$  such that  $h$  satisfies the non-degeneracy condition (2.3) on  $B_k$  or on  $B_{k+m}$ . If no such  $k$  exists, we set  $k_0 := \infty$ . Then for every  $k \in \mathbb{N}$  with  $k < k_0$  condition (2.3) is violated for  $B_k$  and  $\theta_1 B_k = B_{k+m}$ . Therefore, the second alternative of Proposition 2.24 applies and we inductively conclude that

$$\left( \int_{B_{k+1}} |A(\nabla h) - \langle A(\nabla h) \rangle_{B_{k+1}}|^{p'} dx \right)^{\frac{1}{p'}} \leq \theta_2^{k\beta} \left( \int_{B_1} |A(\nabla h) - \langle A(\nabla h) \rangle_{B_1}|^{p'} dx \right)^{\frac{1}{p'}}$$

for all  $1 \leq k < k_0$ . Using Jensen's inequality on the left-hand side and Proposition 2.14 for the right-hand side (note that  $2B_1 \subset B_0$ , since  $\theta_2 < \frac{1}{2}$ ), we conclude the desired estimate for all  $2 \leq k < k_0 + 1$ .

If  $k_0 = \infty$ , the proof is finished. Otherwise, if  $k_0 \in \mathbb{N}_0$ , we are left with showing that for all  $k > k_0$  there holds

$$(2.6) \quad \int_{B_k} |A(\nabla h) - \langle A(\nabla h) \rangle_{B_k}| dx \leq c\theta_2^{(k-k_0)\beta} \int_{B_{k_0}} |A(\nabla h) - \langle A(\nabla h) \rangle_{B_{k_0}}| dx.$$

By construction of  $k_0$ , our solution satisfies the non-degeneracy condition (2.3) on  $B_{k_0}$  or  $B_{k_0+m}$ . In the first case, (2.6) directly follows from Proposition 2.17 and the proof is complete. For the the second case, the same assertion yields that

$$\int_{B_k} |A(\nabla h) - \langle A(\nabla h) \rangle_{B_k}| dx \leq c\theta_2^{(k-(k_0+m))\beta} \int_{B_{k_0+m}} |A(\nabla h) - \langle A(\nabla h) \rangle_{B_{k_0+m}}| dx$$

for every  $k \geq k_0 + m$ . Finally, it remains to note that for each  $\ell \in \{0, 1, \dots, m\}$  we may estimate

$$\int_{B_{k_0+\ell}} |A(\nabla h) - \langle A(\nabla h) \rangle_{B_{k_0+\ell}}| dx \leq c\theta_2^{\ell\beta} \int_{B_{k_0}} |A(\nabla h) - \langle A(\nabla h) \rangle_{B_{k_0}}| dx,$$

since  $\theta_2$  is assumed to be fixed. The combination of the last two bounds then shows (2.6) which completes the proof of Theorem 2.2.  $\square$

**2.6. The case  $1 < p \leq 2$  and proof of Theorem 2.3.** The situation for  $1 < p < 2$  strongly differs from the case  $p \geq 2$ . Let us explain in this subsection what kind of results can be obtained in this situation.

So let us assume here that  $h$  is  $p$ -harmonic on  $\Omega \subset \mathbb{R}^2$  with  $1 < p < 2$ . The optimal regularity of such functions has been studied in detail in [2, 23]. In particular, it has been shown that  $\nabla h \in C_{\text{loc}}^{k, \tilde{\alpha}}(\Omega)$  with

$$k + \tilde{\alpha} = \frac{1}{6} \left( 1 + \frac{1}{p-1} + \sqrt{1 + \frac{14}{p-1} + \frac{1}{(p-1)^2}} \right) =: \eta(p).$$

The optimality of this regularity result has been shown already by Dobrowolski [19, Remark in Section 2]. Expressed in  $A(\nabla h)$  this translates to  $A(\nabla h) \in C_{\text{loc}}^{\ell, \tilde{\beta}}(\Omega)$  with

$$\ell + \tilde{\beta} = \eta(p)(p-1) = \eta(p').$$

Note that  $\ell + \tilde{\beta} \in (0, 1)$  for  $p < 2$  and  $\eta(p) \searrow \frac{1}{3}$ , as  $p \rightarrow \infty$ . In particular, in general  $A(\nabla h) \notin C^1$  for  $p < 2$ . By the same argument it follows that for  $p < 2$  we have  $A(\nabla h) \notin \mathbf{B}_{\varrho, q}^s$  and  $A(\nabla h) \notin \mathbf{F}_{\varrho, q}^s$  for any  $s \in (0, 1)$  such that  $s - \frac{2}{\varrho} > \ell + \tilde{\beta}$ . Therefore, it is not possible to obtain (almost) linear decay estimates of  $A(\nabla h)$  as in Theorem 2.2. Moreover, Theorem 4.1 below cannot hold in full generality for  $p < 2$ , since it fails already for  $F = 0$ .

The natural object to look at for  $1 < p \leq 2$  is  $\nabla u$  rather than  $A(\nabla u)$ . This becomes more clear by duality in the language of differential forms. Indeed, we can use the following nice duality trick from Hamburger [21, Section 5]. Let us assume that  $h$  is  $p$ -harmonic and let us interpret it as a 0-form. Then the 1-form  $\omega := dh$  (which corresponds to  $\nabla h$ ) satisfies

$$\delta A(\omega) = 0, \quad d\omega = 0.$$

Now, if we define the 1-form  $\tau$  by  $*\tau := A(\omega)$  (using that we are in two space dimensions), then

$$\delta A^{-1}(\tau) = 0, \quad d\tau = 0.$$

Since  $d\tau = 0$ , we find a 0-form  $z$  with  $\tau = dz$ . Due to  $A^{-1}(Q) = |Q|^{p'-2}Q$ , we obtain that  $z$  is  $p'$ -harmonic. In particular,  $h$  is  $p$ -harmonic if and only if its conjugate solution  $z$  is  $p'$ -harmonic. Moreover, we have the relation  $*\tau = A(\omega)$  and thus  $\omega = A^{-1}(*\tau)$ . This allows to transfer estimates from  $A(\omega)$  to  $\tau$  and from  $A^{-1}(*\tau)$  to  $\omega$ .

*Proof of Theorem 2.3.* If  $h$  is  $p$ -harmonic with  $p \in (1, 2)$ , then its conjugate  $z$  is  $p'$ -harmonic with  $p' > 2$ . Hence, from Theorem 2.2 we obtain decay estimates for the oscillations of  $A^{-1}(\tau)$ . Using  $\omega = A^{-1}(*\tau) = *A^{-1}(\tau)$  this directly implies decay estimates for  $\omega$ , resp.  $\nabla u$ . In particular, this proves Theorem 2.3.  $\square$

**2.7. Open problems.** We have established in Theorem 2.2 an (almost) linear decay of the oscillation of  $A(\nabla h)$ . This is optimal in the sense that oscillations can never decay faster than linear. However, the limiting case of linear decay unfortunately is excluded by our method of proof. So we ask the following question:

**Question 2.25.** *Is it possible to obtain linear decay of the  $A(\nabla h)$ -oscillations for  $p \geq 2$ , i.e., does Theorem 2.2 also hold with  $\beta = 1$ ?*

**Remark 2.26.** *Let us remark that due to the behavior of  $p$ -harmonic functions  $h$  in the plane a linear decay of  $A(\nabla h)$ -oscillations is only possible for  $p \geq 2$ , see Subsection 2.6. In the case  $p \leq 2$ , the corresponding question would be to obtain linear decay of  $\nabla h$ -oscillations, i.e., Theorem 2.3 with  $\beta = 1$ .*

Parts of the proofs in this section and Section 3 are based on the decay of  $V(\nabla h)$ -oscillations. However, the decay estimate in Lemma 2.22 is non-optimal in the sense that it provides no sharp lower bound for the decay exponent  $\gamma > 0$ . We have used the estimate of [1] in order to prove an (almost) optimal decay of the  $A(\nabla h)$ -oscillations. The same method can be used to prove decay estimates for  $V(\nabla h)$  (in the plane for  $p \geq 2$ ). With the exponent  $\alpha = \alpha(p) > 0$  from Lemma 2.19 we obtain the following estimate:

**Lemma 2.27.** *Let  $p \geq 2$  and let  $h$  be a  $p$ -harmonic function in the plane. Then for every  $\beta \in (0, \frac{\alpha p}{2})$  there exists  $c > 0$  such that*

$$\left( \int_{\theta B} |V(\nabla h) - \langle V(\nabla h) \rangle_{\theta B}|^2 dx \right)^{\frac{1}{2}} \leq c \theta^\beta \left( \int_B |V(\nabla h) - \langle V(\nabla h) \rangle_B|^2 dx \right)^{\frac{1}{2}}$$

for all  $\theta \in (0, 1]$ .

Note that  $\frac{\alpha p}{2} < 1$  for all  $p > 2$ . Moreover, the quantity  $\frac{\alpha p}{2}$  is decreasing in  $p$  and  $\lim_{p \rightarrow \infty} \frac{\alpha p}{2} = \frac{\sqrt{33}-3}{4} \approx 0.6861$ . It is already mentioned in [1] that it is possible to improve  $\alpha(p)$  from Lemma 2.19 a tiny bit by using Young's inequality in the proof. However, this improvement is not enough to raise  $\frac{\alpha p}{2}$  above 1 and the limit at  $p = \infty$  stays the same. So the  $V$ -decay is still strongly sub-linear. However, note that  $\alpha > \frac{1}{p-1}$  for  $p > 2$ , which is the reason why we can still derive (almost) optimal decay for  $A(\nabla h)$ -oscillations.

Nevertheless, the regularity studies in [2, 23] of  $p$ -harmonic functions in the plane indicate a better regularity of  $V(\nabla h)$ , which would allow a linear decay of the  $V(\nabla h)$ -oscillations for all  $1 < p < \infty$ . This is in contrast to the regularity of  $\nabla h$  and  $A(\nabla h)$ . In particular,  $\nabla h \in C^1$  is only possible for  $p \leq 2$  and  $A(\nabla h) \in C^1$  is only possible for  $p \geq 2$ . However, it seems that  $V(\nabla h) \in C^1$  for all  $p$ . We strongly believe that this is also the natural regularity for higher dimensions and vectorial solutions. Therefore we raise the following conjecture:

**Conjecture 2.28.** *For  $d, n \in \mathbb{N}$  and  $1 < p < \infty$  let  $h: \Omega \rightarrow \mathbb{R}^n$  be  $p$ -harmonic on  $\Omega \subset \mathbb{R}^d$ . Then  $V(\nabla h) \in C^1(\Omega)$  and we have a linear decay, i.e.,*

$$\left( \int_{\theta B} |V(\nabla h) - \langle V(\nabla h) \rangle_{\theta B}|^2 dx \right)^{\frac{1}{2}} \leq c \theta \left( \int_B |V(\nabla h) - \langle V(\nabla h) \rangle_B|^2 dx \right)^{\frac{1}{2}}$$

for all balls  $B \subset \Omega$  and every  $\theta \in (0, 1]$ .

Note that  $V(\nabla h) \in C^1$  immediately implies that  $A(\nabla h) \in C^1$  for  $p \geq 2$  and  $\nabla h \in C^1$  for  $p \leq 2$ . In particular, for  $p \geq 2$  it follows that  $\nabla h \in C^{1, \frac{1}{p-1}}$  and therefore  $h \in C^{p'}$  (in the sense of Hölder spaces). Thus the conjecture is stronger than the well known  $p'$ -conjecture, see [1]. In addition, an almost linear decay of the  $V(\nabla h)$ -oscillations would simplify a few steps in Section 3 below.

### 3. OSCILLATION ESTIMATES

In this section we will derive decay estimates for oscillations of the flux  $A(\nabla u)$ . These will be crucial later in deriving Calderón-Zygmund type estimates for  $A(\nabla u)$  in the scale of Besov or Triebel-Lizorkin spaces. The goal of this section is the proof of the following estimate.

**Theorem 3.1.** *Let  $2 \leq p < \infty$  and  $\Omega \subset \mathbb{R}^2$ . For given  $F \in L^{p'}(\Omega)$  let  $u \in W^{1,p}(\Omega)$  be a (scalar) weak solution to*

$$(3.1) \quad -\operatorname{div}(A(\nabla u)) = -\operatorname{div} F \quad \text{in } \Omega.$$

Then for all  $\beta \in (0, 1)$  there exists  $\theta_0 \in (0, 1)$  and  $c = c(\beta, \theta_0) > 0$  such that for all balls  $B \subset \Omega$  there holds

$$\begin{aligned} & \left( \int_{\theta_0 B} |A(\nabla u) - \langle A(\nabla u) \rangle_{\theta_0 B}|^{p'} dx \right)^{\frac{1}{p'}} \\ & \leq \theta_0^\beta \left( \int_B |A(\nabla u) - \langle A(\nabla u) \rangle_B|^{p'} dx \right)^{\frac{1}{p'}} + c \left( \int_B |F - \langle F \rangle_B|^{p'} dx \right)^{\frac{1}{p'}}. \end{aligned}$$

**Remark 3.2.** *In the case of higher dimensions and vectorial solutions we get the same oscillation decay estimate but with  $\beta$  restricted to  $\beta \in (0, \beta_0)$ , where  $\beta_0 \in (0, 1)$  is some (unknown) small number. The reason is the worse decay estimate for  $p$ -harmonic functions in this more general situation, see Remark 2.4. In fact, our oscillation estimates hold exactly in the same range as the decay estimate for  $p$ -harmonic functions.*

Theorem 3.1 shows that the oscillation of  $A(\nabla u)$  decreases for some fixed(!) reduction of the radius by a factor of  $\theta_0$ . We can iterate the estimate to obtain an oscillation decay for arbitrary reductions  $\theta \in (0, 1)$ . However, to formulate this it is useful to introduce the following short notations on oscillations.

Let  $B_t(x)$  denote the ball of radius  $t > 0$  centered at  $x$ . Then for  $g \in L^w_{\text{loc}}(\mathbb{R}^d)$ ,  $w \geq 1$ , we define its (zero order) oscillation by

$$\text{osc}_w g(x, t) := \left( \int_{B_t(x)} |g(y) - \langle g \rangle_{B_t(x)}|^w dy \right)^{\frac{1}{w}}.$$

Note that in this definition it is possible to replace the mean by an infimum over all constants, which gives rise to an equivalent expression, see Lemma 2.10.

**Theorem 3.3.** *Let  $u$ ,  $p$ ,  $F$ , and  $\beta$  be as in Theorem 3.1. Then there exists  $c > 0$  such that for all  $\theta \in (0, 1)$  and all balls  $B_t(x) \subset \Omega$  there holds*

$$\text{osc}_{p'} A(\nabla u)(x, \theta t) \leq c \theta^\beta \text{osc}_{p'} A(\nabla u)(x, t) + c \theta^\beta \int_\theta^1 \lambda^{-\beta} \text{osc}_{p'} F(x, \lambda t) \frac{d\lambda}{\lambda}.$$

Both theorems will be proven in Subsection 3.4.

**3.1. Non-linear comparison.** In the proof of our oscillation estimates, we will need to compare the function  $u$  locally with the  $p$ -harmonic function  $h$  that solves

$$(3.2) \quad \begin{aligned} -\text{div}(A(\nabla h)) &= 0 && \text{in } B, \\ h &= u && \text{on } \partial B. \end{aligned}$$

The basic idea is to transfer the decay estimate of  $V(\nabla h)$  to  $V(\nabla u)$ , resp.  $A(\nabla h)$  to  $A(\nabla u)$ , by using the following comparison result.

**Lemma 3.4** (Non-linear comparison). *Let  $h$  be the solution of (3.2). Then*

$$\int_B |V(\nabla u) - V(\nabla h)|^2 dx \lesssim \int_B \varphi_{|A(\nabla u)|}^*(|F - \langle F \rangle_B|) dx.$$

*Proof.* We take the difference of the equations for  $u$  and  $h$  and test it with  $u - h$  scaled by  $|B|^{-1}$ . So for arbitrarily small  $\delta > 0$  we obtain

$$\begin{aligned} \int_B |V(\nabla u) - V(\nabla h)|^2 dx &\lesssim \int_B (A(\nabla u) - A(\nabla h)) \cdot (\nabla u - \nabla h) dx \\ &\leq \int_B |F - \langle F \rangle_B| |\nabla u - \nabla h| dx \\ &\leq c_\delta \int_B \varphi_{|A(\nabla u)|}^*(|F - \langle F \rangle_B|) dx + \delta \int_B \varphi_{|\nabla u|}(|\nabla u - \nabla h|) dx \\ &\lesssim c_\delta \int_B \varphi_{|A(\nabla u)|}^*(|F - \langle F \rangle_B|) dx + \delta \int_B |V(\nabla u) - V(\nabla h)|^2 dx, \end{aligned}$$

where we have used Lemma 2.7, as well as  $(\varphi_{|\nabla u|})^* \approx \varphi_{|A(\nabla u)|}^*$ . Now we absorb the last integral to prove the claim.  $\square$

In the following we need an estimate of reverse Hölder type from [6, Corollary 2.4] which is also contained in the proof of [15, Corollary 3.5].

**Lemma 3.5** ([6, Corollary 2.4]). *Let  $u$  solve (3.1). Then for all vectors  $Q$  we have*

$$\begin{aligned} & \int_B |V(\nabla u) - V(Q)|^2 dx \\ & \leq \varphi_{|A(Q)|}^* \left( \int_{2B} |A(\nabla u) - A(Q)| dx \right) + c \int_{2B} \varphi_{|A(Q)|}^* (|F - \langle F \rangle_{2B}|) dx. \end{aligned}$$

Now the decay assertion for  $V(\nabla h)$  in Lemma 2.22 provides us with a preliminary decay estimate for  $V(\nabla u)$ . However note that the decay exponent is far from being optimal. Anyhow, we need this decay estimate to control our final oscillation on a small subset.

**Lemma 3.6.** *Let  $\gamma > 0$  be as in Lemma 2.22. Then there exists  $c = c(\gamma) > 0$  such that we have the following decay estimate:*

$$\begin{aligned} & \int_{\theta B} |V(\nabla u) - \langle V(\nabla u) \rangle_{\theta B}|^2 dx \\ & \leq c \theta^{2\gamma} \int_{2B} |A(\nabla u) - \langle A(\nabla u) \rangle_{2B}|^{p'} dx + c \theta^{-d} \int_{2B} |F - \langle F \rangle_{2B}|^{p'} dx. \end{aligned}$$

*Proof.* Let  $h$  be the solution of (3.2). We estimate

$$\begin{aligned} I_\theta & := \int_{\theta B} |V(\nabla u) - \langle V(\nabla u) \rangle_{\theta B}|^2 dx \\ & \lesssim \int_{\theta B} |V(\nabla h) - \langle V(\nabla h) \rangle_{\theta B}|^2 dx + \int_{\theta B} |V(\nabla u) - V(\nabla h)|^2 dx \end{aligned}$$

and use the decay estimate for  $V(\nabla h)$ , see Lemma 2.22, together with  $\theta^{2\gamma} < \theta^{-d}$  to conclude that

$$\begin{aligned} I_\theta & \lesssim \theta^{2\gamma} \int_B |V(\nabla h) - \langle V(\nabla h) \rangle_B|^2 dx + \theta^{-d} \int_B |V(\nabla u) - V(\nabla h)|^2 dx \\ & \lesssim \theta^{2\gamma} \int_B |V(\nabla u) - \langle V(\nabla u) \rangle_B|^2 dx + \theta^{-d} \int_B |V(\nabla u) - V(\nabla h)|^2 dx. \end{aligned}$$

Now, Lemma 2.11 and Lemma 3.4, imply

$$I_\theta \lesssim \theta^{2\gamma} \int_B |V(\nabla u) - V(\langle \nabla u \rangle_B^A)|^2 dx + \theta^{-d} \int_B \varphi_{|A(\nabla u)|}^* (|F - \langle F \rangle_B|) dx.$$

For the first integral we can employ Lemma 3.5 with  $Q := \langle \nabla u \rangle_B^A$  and Corollary 2.8 to obtain

$$\begin{aligned} & \int_B |V(\nabla u) - V(\langle \nabla u \rangle_B^A)|^2 dx \\ & \leq \varphi_{|A(Q)|}^* \left( \int_{2B} |A(\nabla u) - \langle A(\nabla u) \rangle_B| dx \right) + c \int_{2B} \varphi_{|A(Q)|}^* (|F - \langle F \rangle_{2B}|) dx \\ & \lesssim \int_{2B} |A(\nabla u) - \langle A(\nabla u) \rangle_B|^{p'} dx + \int_{2B} |F - \langle F \rangle_{2B}|^{p'} dx \\ & \lesssim \int_{2B} |A(\nabla u) - \langle A(\nabla u) \rangle_{2B}|^{p'} dx + \theta^{-2\gamma-d} \int_{2B} |F - \langle F \rangle_{2B}|^{p'} dx. \end{aligned}$$

Similarly the second integral can be estimated by

$$\int_B \varphi_{|A(\nabla u)|}^* (|F - \langle F \rangle_B|) dx \lesssim \int_B |F - \langle F \rangle_B|^{p'} dx \lesssim \int_{2B} |F - \langle F \rangle_{2B}|^{p'} dx.$$

Hence, combining the last two bounds shows the claim.  $\square$

**3.2. Degenerate case.** Let us begin with the degenerate case. In particular, we assume that

$$(3.3) \quad \int_B |V(\nabla u) - \langle V(\nabla u) \rangle_B|^2 dx > \varepsilon_{\text{DG}} \int_B |V(\nabla u)|^2 dx.$$

The parameter  $\varepsilon_{\text{DG}} > 0$  is fixed in this section. The specific value of  $\varepsilon_{\text{DG}}$  will be determined later by the non-degenerate case.

We are now prepared to prove the desired  $A$ -decay estimate.



**Proposition 3.7.** *Let  $\beta \in (0, 1)$ . Then there exists a constant  $c = c(\beta) > 0$  such that for every  $\theta, \varepsilon_{\text{DG}} \in (0, 1)$  we have the following decay estimate on balls  $B$  with (3.3):*

$$\begin{aligned} \left( \int_{\theta B} |A(\nabla u) - \langle A(\nabla u) \rangle_{\theta B}|^{p'} dx \right)^{\frac{1}{p'}} &\leq c \theta^\beta \left( \int_{2B} |A(\nabla u) - \langle A(\nabla u) \rangle_{2B}|^{p'} dx \right)^{\frac{1}{p'}} \\ &\quad + c \varepsilon_{\text{DG}}^{\frac{-(p-1)^2}{p}} \theta^{-(p-1)(\beta+d)} \left( \int_{2B} |F - \langle F \rangle_{2B}|^{p'} dx \right)^{\frac{1}{p'}}. \end{aligned}$$

*Proof.* Let  $h$  be the solution of (3.2). Similar to the proof of Lemma 3.6 we estimate

$$\begin{aligned} I_\theta &:= \int_{\theta B} |A(\nabla u) - \langle A(\nabla u) \rangle_{\theta B}|^{p'} dx \\ &\lesssim \int_{\theta B} |A(\nabla h) - \langle A(\nabla h) \rangle_{\theta B}|^{p'} dx + \int_{\theta B} |A(\nabla u) - A(\nabla h)|^{p'} dx, \end{aligned}$$

where this time the decay estimate for  $A(\nabla h)$ , see Proposition 2.15, implies

$$I_\theta \lesssim \theta^{p'\beta} \int_B |A(\nabla h) - \langle A(\nabla h) \rangle_B|^{p'} dx + \theta^{-d} \int_B |A(\nabla u) - A(\nabla h)|^{p'} dx.$$

Now we shall show that the second integral is bounded by

$$\begin{aligned} R &:= \int_B |A(\nabla u) - A(\nabla h)|^{p'} dx \\ &\lesssim \theta^{p'\beta+d} \int_B |A(\nabla u) - \langle A(\nabla u) \rangle_B|^{p'} dx + \varepsilon_{\text{DG}}^{1-p} \theta^{-p\beta+d(1-p)} \int_B |F - \langle F \rangle_B|^{p'} dx, \end{aligned}$$

because then the claim follows by replacing  $B$  by  $2B$  in all occurring averages. To this end, we employ a shift-change (Lemma 2.9 applied to  $P = 0$  and  $Q = \nabla u$ ), which shows that for  $\lambda \in (0, 1]$  (to be specified later) there holds

$$\begin{aligned} R &\lesssim \int_B \varphi_0^*(|A(\nabla u) - A(\nabla h)|) dx \\ &\lesssim \lambda \int_B |V(\nabla u)|^2 dx + \lambda^{1-p} \int_B \varphi_{|A(\nabla u)|}^*(|A(\nabla u) - A(\nabla h)|) dx. \end{aligned}$$

Here the first integral can be bounded by using the degeneracy condition (3.3), Lemmata 2.7 and 2.11, as well as Corollary 2.8 (using  $p \geq 2$ ) which gives

$$\begin{aligned} \int_B |V(\nabla u)|^2 dx &< \varepsilon_{\text{DG}}^{-1} \int_B |V(\nabla u) - \langle V(\nabla u) \rangle_B|^2 dx \\ &\approx \varepsilon_{\text{DG}}^{-1} \int_B |V(\nabla u) - V(\langle \nabla u \rangle_B^A)|^2 dx \\ &\lesssim \varepsilon_{\text{DG}}^{-1} \int_B |A(\nabla u) - \langle A(\nabla u) \rangle_B|^{p'} dx. \end{aligned}$$

In addition, the other integral can be estimated by Lemma 2.7, non-linear comparison (Lemma 3.4) and Corollary 2.8 again such that we obtain

$$\begin{aligned} \int_B \varphi_{|A(\nabla u)|}^*(|A(\nabla u) - A(\nabla h)|) dx &\lesssim \int_B |V(\nabla u) - V(\nabla h)|^2 dx \\ &\lesssim \int_B \varphi_{|A(\nabla u)|}^*(|F - \langle F \rangle_B|) dx \\ &\lesssim \int_B |F - \langle F \rangle_B|^{p'} dx. \end{aligned}$$

Hence, we have shown that

$$R \lesssim \frac{\lambda}{\varepsilon_{\text{DG}}} \int_B |A(\nabla u) - \langle A(\nabla u) \rangle_B|^{p'} dx + \lambda^{1-p} \int_B |F - \langle F \rangle_B|^{p'} dx.$$

Since  $p'(1-p) = -p$  choosing  $\lambda := \varepsilon_{\text{DG}} \theta^{p'\beta+d}$  now yields the claimed estimate on  $R$  and thus the proof is complete.  $\square$

**3.3. Non-degenerate case.** Let us now turn to the non-degenerate case. In particular, we will assume that  $u$  satisfies the following non-degeneracy condition on  $B$

$$(3.4) \quad \int_B |V(\nabla u) - \langle V(\nabla u) \rangle_B|^2 dx \leq \varepsilon_{\text{DG}} \int_B |V(\nabla u)|^2 dx.$$

Unfortunately, we cannot proceed as in the degenerate case and compare  $u$  with a  $p$ -harmonic function  $h$ . The reason is a technical one, namely that the shift-changes cannot be controlled by means of oscillations.

However, the non-degeneracy condition ensures that  $V(\nabla u)$  is in some sense close to the constant  $\langle V(\nabla u) \rangle_B$ . This implies that  $\nabla u$  is close to  $\langle \nabla u \rangle_B^A$ . Hence, the system behaves approximately like a linear one with constant coefficients. In particular, this argument works best on the set where  $|\nabla u - \langle \nabla u \rangle_B^A| \ll |\langle \nabla u \rangle_B^A|$ . The non-degeneracy condition however is only in the integral sense, so there is a small set of points that fail this condition. It turns out that we can control the critical terms on this set by the (non-optimal) decay estimates of Lemma 3.6. This is done in Lemma 3.10 below. On the remaining “nice” set, we will estimate the  $A$ -oscillation by using an approximation by a linear system with constant coefficients, see Lemma 3.12.

Before we get to Lemma 3.10, we need a few auxiliary results on averages. The subsequent two lemmata follow the spirit of [6, Lemma 2.12].

**Lemma 3.8.** *There exists a constant  $\varepsilon_0$  such that if  $u$  satisfies the non-degeneracy condition (3.4) on  $B$  with  $\varepsilon_{\text{DG}} \leq \varepsilon_0$ , then*

$$(3.5) \quad \frac{1}{2} \max \{ |\langle \nabla u \rangle_B|, |\langle \nabla u \rangle_B^A| \} \leq |\langle \nabla u \rangle_B^V| \leq 2 \min \{ |\langle \nabla u \rangle_B|, |\langle \nabla u \rangle_B^A| \},$$

and

$$(3.6) \quad \max \{ |\langle \nabla u \rangle_B - \langle \nabla u \rangle_B^V|, |\langle \nabla u \rangle_B^A - \langle \nabla u \rangle_B^V| \} \leq c \sqrt{\varepsilon_{\text{DG}}} |\langle \nabla u \rangle_B^V|.$$

*Proof.* Using (3.4) we estimate

$$\begin{aligned} \int_B |V(\nabla u) - \langle V(\nabla u) \rangle_B|^2 dx &\leq \varepsilon_{\text{DG}} \int_B |V(\nabla u)|^2 dx \\ &\leq 2\varepsilon_{\text{DG}} \int_B |V(\nabla u) - \langle V(\nabla u) \rangle_B|^2 dx + 2\varepsilon_{\text{DG}} |\langle V(\nabla u) \rangle_B|^2. \end{aligned}$$

For  $\varepsilon_{\text{DG}} \leq \varepsilon_0 \leq \frac{1}{4}$  we can absorb the first term on the right-hand side and obtain

$$(3.7) \quad \int_B |V(\nabla u) - \langle V(\nabla u) \rangle_B|^2 dx \leq 4\varepsilon_{\text{DG}} |\langle V(\nabla u) \rangle_B|^2 = 4\varepsilon_{\text{DG}} |V(\langle \nabla u \rangle_B^V)|^2.$$

Now let  $Q \in \{ \langle \nabla u \rangle_B, \langle \nabla u \rangle_B^A \}$ . Then we can use Lemma 2.11 and (3.7) to derive

$$\begin{aligned} |V(\langle \nabla u \rangle_B^V) - V(Q)|^2 &= |\langle V(\nabla u) \rangle_B - V(Q)|^2 \\ &\leq \int_B |V(\nabla u) - V(Q)|^2 dx \\ &\leq c \int_B |V(\nabla u) - \langle V(\nabla u) \rangle_B|^2 dx \\ (3.8) \quad &\leq c\varepsilon_{\text{DG}} |V(\langle \nabla u \rangle_B^V)|^2 \end{aligned}$$

and hence

$$(1 - c\varepsilon_{\text{DG}}^{1/2})^2 |V(\langle \nabla u \rangle_B^V)|^2 \leq |V(Q)|^2 \leq (1 + c\varepsilon_{\text{DG}}^{1/2})^2 |V(\langle \nabla u \rangle_B^V)|^2.$$

Now we choose  $\varepsilon_0 \geq \varepsilon_{\text{DG}}$  small enough and use that  $|V(Q)| = |Q|^{p/2}$  to conclude

$$\frac{1}{2} |\langle \nabla u \rangle_B^V| \leq |Q| \leq 2 |\langle \nabla u \rangle_B^V|, \quad Q \in \{ \langle \nabla u \rangle_B, \langle \nabla u \rangle_B^A \},$$

which shows (3.5).

It remains to prove (3.6). To this end, let  $P := \langle \nabla u \rangle_B^V$ . Since  $p \geq 2$ , it then follows from Lemma 2.7 and (3.8) that

$$|P|^{p-2} |P - Q|^2 \leq (|P| + |Q|)^{p-2} |P - Q|^2 \leq c \varepsilon_{\text{DG}} |V(P)|^2 = c \varepsilon_{\text{DG}} |P|^p,$$

i.e.,

$$|P - Q| \leq c \sqrt{\varepsilon_{\text{DG}}} |P|$$

which completes the proof.  $\square$

If  $\varepsilon_{\text{DG}}$  is small enough, our non-degeneracy condition passes over from  $B$  to some sub-balls:

**Lemma 3.9.** *For all  $\tau \in (0, 1)$  there exists  $\varepsilon = \varepsilon(\tau) > 0$  with the following property: If  $u$  satisfies the non-degeneracy condition (3.4) on  $B$  with some  $\varepsilon_{\text{DG}} \leq \varepsilon$ , then  $u$  also satisfies (3.4) on  $\tau B$  with  $\varepsilon_{\text{DG}}$  replaced by  $16 \tau^{-d} \varepsilon_{\text{DG}}$  and we have*

$$(3.9) \quad \begin{aligned} \frac{1}{2} |\langle V(\nabla u) \rangle_B| &\leq |\langle V(\nabla u) \rangle_{\tau B}| \leq 2 |\langle V(\nabla u) \rangle_B|, \\ \frac{1}{2} |\langle \nabla u \rangle_B^V| &\leq |\langle \nabla u \rangle_{\tau B}^V| \leq 2 |\langle \nabla u \rangle_B^V|. \end{aligned}$$

*Proof.* Let us show (3.9) first. For this purpose we use (3.4) on  $B$  to estimate

$$\begin{aligned} |\langle V(\nabla u) \rangle_{\tau B} - \langle V(\nabla u) \rangle_B|^2 dx &\leq \int_{\tau B} |V(\nabla u) - \langle V(\nabla u) \rangle_B|^2 dx \\ &\leq \tau^{-d} \int_B |V(\nabla u) - \langle V(\nabla u) \rangle_B|^2 dx \\ &\leq \tau^{-d} \varepsilon_{\text{DG}} |\langle V(\nabla u) \rangle_B|^2. \end{aligned}$$

Now fix  $\delta \in (0, \frac{1}{2})$  small enough such that  $(1-\delta)^{\frac{2}{p}} \geq \frac{1}{2}$  and  $(1+\delta)^{\frac{2}{p}} \leq 2$ . Then, for small  $\varepsilon \geq \varepsilon_{\text{DG}} > 0$ , we have  $\tau^{-d} \varepsilon_{\text{DG}} \leq \delta^2$  and therefore

$$|\langle V(\nabla u) \rangle_{\tau B} - \langle V(\nabla u) \rangle_B| \leq \delta |\langle V(\nabla u) \rangle_B|,$$

i.e.,

$$(1 - \delta) |\langle V(\nabla u) \rangle_B| \leq |\langle V(\nabla u) \rangle_{\tau B}| \leq (1 + \delta) |\langle V(\nabla u) \rangle_B|.$$

Thus, our choice of  $\delta$  and the fact that  $|\langle V(\nabla u) \rangle_B| = |\langle \nabla u \rangle_B^V|^{\frac{p}{2}}$  proves (3.9).

Further, we may estimate

$$\begin{aligned} \int_{\tau B} |V(\nabla u) - \langle V(\nabla u) \rangle_{\tau B}|^2 dx &\leq \int_{\tau B} |V(\nabla u) - \langle V(\nabla u) \rangle_B|^2 dx \\ &\leq \tau^{-d} \int_B |V(\nabla u) - \langle V(\nabla u) \rangle_B|^2 dx. \end{aligned}$$

We can additionally assume that  $\varepsilon$  is so small that (3.7) from the proof of Lemma 3.8 holds true. Together with the first part of (3.9) this yields

$$\begin{aligned} \int_{\tau B} |V(\nabla u) - \langle V(\nabla u) \rangle_{\tau B}|^2 dx &\leq 4 \tau^{-d} \varepsilon_{\text{DG}} |\langle V(\nabla u) \rangle_B|^2 \\ &\leq 16 \tau^{-d} \varepsilon_{\text{DG}} |\langle V(\nabla u) \rangle_{\tau B}|^2 \\ &\leq 16 \tau^{-d} \varepsilon_{\text{DG}} \int_{\tau B} |V(\nabla u)|^2 dx \end{aligned}$$

which completes the proof.  $\square$

The following lemma is an adaptation of [6, Lemma 2.19].

**Lemma 3.10.** *Let  $\sigma, \tau \in (0, \frac{1}{4})$ . Then there exists  $\varepsilon = \varepsilon(\sigma, \tau) > 0$  such that if  $u$  satisfies the non-degeneracy condition (3.4) on  $B$  with  $\varepsilon_{\text{DG}} \leq \varepsilon$ , then*

$$\begin{aligned} & \int_B |A(\nabla u) - \langle A(\nabla u) \rangle_B|^{p'} \chi_{\{|\nabla u - \langle \nabla u \rangle_B^A| \geq \sigma |\langle \nabla u \rangle_B^A\}} dx \\ & \leq c \sigma^{-2p} \tau^{2\gamma} \left( \int_{2B} |A(\nabla u) - \langle A(\nabla u) \rangle_{2B}|^{p'} dx + \tau^{-d-2\gamma} \int_{2B} |F - \langle F \rangle_{2B}|^{p'} dx \right) \end{aligned}$$

with  $\gamma$  from Lemma 2.22.

*Proof.* Let  $\sigma, \tau \in (0, \frac{1}{4})$ . Then it is possible to cover  $B$  by a locally finite set of balls  $\tau B_j$ , where the  $B_j$  are translates of  $B$  with centers within  $B$ . In particular,  $B \subset \bigcup_j (\tau B_j) \subset 2B$  and  $B_j \subset 2B$ . We define

$$I_j := \int_{\tau B_j} |A(\nabla u) - \langle A(\nabla u) \rangle_B|^{p'} \chi_{E_\sigma} dx$$

with  $E_\sigma := \{x \mid |\nabla u - \langle \nabla u \rangle_B^A| \geq \sigma |\langle \nabla u \rangle_B^A|\}$ . Then

$$I := \int_B |A(\nabla u) - \langle A(\nabla u) \rangle_B|^{p'} \chi_{\{|\nabla u - \langle \nabla u \rangle_B^A| \geq \sigma |\langle \nabla u \rangle_B^A\}} dx \leq \sum_j \frac{|\tau B_j|}{|B|} I_j.$$

If  $\varepsilon = \varepsilon(\tau)$  is small enough, then according to Lemma 3.9  $u$  also satisfies the non-degeneracy condition (3.4) w.r.t. the ball  $\tau B_j$ . Now, Lemma 3.8 for  $B$  and  $\tau B_j$  and Lemma 3.9 imply that

$$(3.10) \quad \begin{aligned} |\langle \nabla u \rangle_B^A| &\leq 2 |\langle \nabla u \rangle_B^V| \leq 4 |\langle \nabla u \rangle_{\tau B_j}^V| \leq 8 |\langle \nabla u \rangle_{\tau B_j}^A|, \\ |\langle \nabla u \rangle_{\tau B_j}^A| &\leq 2 |\langle \nabla u \rangle_{\tau B_j}^V| \leq 4 |\langle \nabla u \rangle_B^V| \leq 8 |\langle \nabla u \rangle_B^A|. \end{aligned}$$

Therefore, on the set  $\tau B_j$  we can estimate

$$(3.11) \quad \begin{aligned} |\nabla u - \langle \nabla u \rangle_B^A| \chi_{E_\sigma} &\leq (|\nabla u - \langle \nabla u \rangle_{\tau B_j}^A| + |\langle \nabla u \rangle_B^A| + |\langle \nabla u \rangle_{\tau B_j}^A|) \chi_{E_\sigma} \\ &\leq (|\nabla u - \langle \nabla u \rangle_{\tau B_j}^A| + 9 |\langle \nabla u \rangle_B^A|) \chi_{E_\sigma} \\ &\leq c \sigma^{-1} |\nabla u - \langle \nabla u \rangle_{\tau B_j}^A| \chi_{E_\sigma} \end{aligned}$$

since  $\sigma < 1$ . Moreover, we can employ Lemma 2.7 (with  $P := \nabla u$  and  $Q := \langle \nabla u \rangle_B^A$  and  $p \geq 2$ ) to obtain

$$\begin{aligned} |A(\nabla u) - \langle A(\nabla u) \rangle_B|^{p'} \chi_{E_\sigma} &\leq c \left( (|\langle \nabla u \rangle_B^A| + |\nabla u - \langle \nabla u \rangle_B^A|)^{p-2} |\nabla u - \langle \nabla u \rangle_B^A| \right)^{p'} \chi_{E_\sigma} \\ &\leq c |V(\nabla u) - V(\langle \nabla u \rangle_B^A)|^2 \left( \frac{|\langle \nabla u \rangle_B^A| + |\nabla u - \langle \nabla u \rangle_B^A|}{|\nabla u - \langle \nabla u \rangle_B^A|} \right)^{2-p'} \chi_{E_\sigma} \\ &\leq c \sigma^{p'-2} |V(\nabla u) - V(\langle \nabla u \rangle_B^A)|^2 \chi_{E_\sigma}. \end{aligned}$$

Applying Lemma 2.7 once more, together with (3.10), (3.11) and the fact that  $p' - 2 - p > -2p$  this yields

$$\begin{aligned} |A(\nabla u) - \langle A(\nabla u) \rangle_B|^{p'} \chi_{E_\sigma} &\leq c \sigma^{p'-2} \varphi_{|\langle \nabla u \rangle_B^A|} (|\nabla u - \langle \nabla u \rangle_B^A|) \chi_{E_\sigma} \\ &\leq c \sigma^{p'-2} \varphi_{|\langle \nabla u \rangle_{\tau B_j}^A|} (c \sigma^{-1} |\nabla u - \langle \nabla u \rangle_{\tau B_j}^A|) \chi_{E_\sigma} \\ &\leq c \sigma^{-2p} \varphi_{|\langle \nabla u \rangle_{\tau B_j}^A|} (|\nabla u - \langle \nabla u \rangle_{\tau B_j}^A|) \chi_{E_\sigma} \\ &\leq c \sigma^{-2p} |V(\nabla u) - V(\langle \nabla u \rangle_{\tau B_j}^A)|^2 \end{aligned}$$

on every  $\tau B_j$  such that

$$I \leq c \sigma^{-2p} \sum_j \frac{|\tau B_j|}{|B|} \int_{\tau B_j} |V(\nabla u) - V(\langle \nabla u \rangle_{\tau B_j}^A)|^2 dx.$$

For these local integrals it follows from Lemma 3.6 (applied for  $\frac{1}{2}B_j$ ),  $|B_j| = |B|$ , and  $B_j \subset 2B$  that

$$\begin{aligned} & \int_{\tau B_j} |V(\nabla u) - \langle V(\nabla u) \rangle_{\tau B}|^2 dx \\ & \leq c(2\tau)^{2\gamma} \int_{B_j} |A(\nabla u) - \langle A(\nabla u) \rangle_{B_j}|^{p'} dx + c(2\tau)^{-d} \int_{B_j} |F - \langle F \rangle_{B_j}|^{p'} dx. \\ & \leq c\tau^{2\gamma} \int_{2B} |A(\nabla u) - \langle A(\nabla u) \rangle_{2B}|^{p'} dx + c\tau^{-d} \int_{2B} |F - \langle F \rangle_{2B}|^{p'} dx. \end{aligned}$$

This together with the previous estimate and the covering properties of the  $\tau B_j$  proves

$$\begin{aligned} I & \lesssim \sigma^{-2p} \sum_j \frac{|\tau B_j|}{|B|} \left( \tau^{2\gamma} \int_{2B} |A(\nabla u) - \langle A(\nabla u) \rangle_{2B}|^{p'} dx + \tau^{-d} \int_{2B} |F - \langle F \rangle_{2B}|^{p'} dx \right) \\ & \leq c\sigma^{-2p} \tau^{2\gamma} \left( \int_{2B} |A(\nabla u) - \langle A(\nabla u) \rangle_{2B}|^{p'} dx + \tau^{-d-2\gamma} \int_{2B} |F - \langle F \rangle_{2B}|^{p'} dx \right), \end{aligned}$$

as claimed.  $\square$

Sometimes it is useful to apply Lemma 3.10 with a different kind of indicator set, namely with the  $A$ -mean value  $\langle \nabla u \rangle_B^A$  replaces by the standard mean value  $\langle \nabla u \rangle_B$  on  $B$ . The following lemma shows that the two cases are the same up to a possible change of the constant  $\sigma$ .

**Lemma 3.11.** *For all  $\sigma > 0$  there exists  $\varepsilon = \varepsilon(\sigma) > 0$  such that if  $u$  satisfies the non-degeneracy condition (3.4) on  $B$  with  $\varepsilon_{\text{DG}} \leq \varepsilon$ , then there holds*

$$\{x \mid |\nabla u - \langle \nabla u \rangle_B| \geq \sigma |\langle \nabla u \rangle_B|\} \subseteq \left\{x \mid |\nabla u - \langle \nabla u \rangle_B^A| \geq \frac{\sigma}{8} |\langle \nabla u \rangle_B^A|\right\}.$$

*Proof.* For each  $x$  with  $|\nabla u(x) - \langle \nabla u \rangle_B| \geq \sigma |\langle \nabla u \rangle_B|$  and small enough  $\varepsilon$  we estimate with Lemma 3.8

$$\begin{aligned} |\nabla u(x) - \langle \nabla u \rangle_B^A| & \geq |\nabla u(x) - \langle \nabla u \rangle_B| - |\langle \nabla u \rangle_B^A - \langle \nabla u \rangle_B| \\ & \geq \sigma |\langle \nabla u \rangle_B| - c\sqrt{\varepsilon_{\text{DG}}} |\langle \nabla u \rangle_B^V|. \\ & \geq \frac{\sigma}{4} |\langle \nabla u \rangle_B^A| - c\sqrt{\varepsilon} |\langle \nabla u \rangle_B^A|. \end{aligned}$$

So for  $c\sqrt{\varepsilon} \leq \frac{\sigma}{8}$  we get

$$|\nabla u(x) - \langle \nabla u \rangle_B^A| \geq \frac{\sigma}{8} |\langle \nabla u \rangle_B^A|$$

which proves the claim.  $\square$

In order to proceed towards the desired linear comparison result, let  $z$  denote the solution of the following linear system with constant coefficients:

$$(3.12) \quad \begin{aligned} -\operatorname{div}((DA)(\langle \nabla u \rangle_B^A)\nabla z) &= 0 && \text{in } B, \\ z &= u && \text{on } \partial B. \end{aligned}$$

We know from linear theory that there exists some constant  $c > 0$  such that for any  $\theta \in (0, 1)$  we have

$$(3.13) \quad \sup_{x, x' \in \theta B} |\nabla z(x) - \nabla z(x')| \leq c\theta \int_B |\nabla z - \langle \nabla z \rangle_B| dy.$$

For vectors  $P, Q \in \mathbb{R}^2$  let us define

$$H(P, Q) := A(P) - A(Q) - (DA)(Q)(P - Q).$$

Then we can use our original system (1.1),

$$-\operatorname{div}(A(\nabla u)) = -\operatorname{div} F,$$

to conclude

$$(3.14) \quad -\operatorname{div}((DA)(\langle \nabla u \rangle_B^A)\nabla(u - z)) = \operatorname{div}(H(\nabla u, \langle \nabla u \rangle_B^A)) - \operatorname{div}(F - \langle F \rangle_B).$$

In particular, the function  $w := u - z$  satisfies

$$\begin{aligned} -\operatorname{div}((DA)(\langle \nabla u \rangle_B^A) \nabla w) &= \operatorname{div}(H(\nabla u, \langle \nabla u \rangle_B^A)) - \operatorname{div}(F - \langle F \rangle_B) && \text{in } B, \\ w &= 0 && \text{on } \partial B. \end{aligned}$$

It follows from Lemma 2.7 for  $P := Q + t\xi$  with  $t \rightarrow 0$  and arbitrarily fixed  $Q, \xi \in \mathbb{R}^2$  that the constant matrix  $(DA)(Q)$  satisfies

$$(3.15) \quad c|Q|^{p-2}|\xi|^2 \leq ((DA)(Q)\xi) \cdot \xi \leq C|Q|^{p-2}|\xi|^2.$$

As in [17] and [6] we get the following comparison estimate.

**Lemma 3.12** (Linear Comparison). *Let  $z$  be the solution of (3.12), then*

$$|\langle \nabla u \rangle_B^A|^{(p-2)p'} \int_B |\nabla u - \nabla z|^{p'} dx \leq c \int_B |H(\nabla u, \langle \nabla u \rangle_B^A)|^{p'} dx + c \int_B |F - \langle F \rangle_B|^{p'} dx.$$

*Proof.* If  $w := u - z$ , then it is the solution to a linear system of the form

$$\begin{aligned} -\operatorname{div}(\mathcal{B}\nabla w) &= -\operatorname{div} G && \text{in } B, \\ w &= 0 && \text{on } \partial B, \end{aligned}$$

where we put  $\mathcal{B} := (DA)(\langle \nabla u \rangle_B^A)$  and  $G := -H(\nabla u, \langle \nabla u \rangle_B^A) + (F - \langle F \rangle_B)$ . Due to (3.15) with  $Q = \langle \nabla u \rangle_B^A$ , we have the ellipticity condition

$$c|\langle \nabla u \rangle_B^A|^{p-2}|\xi|^2 \leq \mathcal{B}\xi \cdot \xi \leq C|\langle \nabla u \rangle_B^A|^{p-2}|\xi|^2, \quad \xi \in \mathbb{R}^2.$$

Therefore we can apply the classical  $L^{p'}$ -regularity result for systems with constant coefficients, see [20, Lemma 2], to conclude

$$\| |\langle \nabla u \rangle_B^A|^{p-2} \nabla w \|_{L^{p'}(B)} \leq c \|G\|_{L^{p'}(B)}.$$

Thus, the definitions of  $w$  and  $G$  prove the claim.  $\square$

In order to estimate the  $H$ -term in Lemma 3.12 we need the following Lemma.

**Lemma 3.13.** *If  $p \geq 2$ , then we have*

$$|H(P, Q)| \leq c|A(P) - A(Q)| \frac{|P - Q|}{|Q| + |P - Q|}$$

for all  $P, Q \in \mathbb{R}^2$ .

*Proof.* We have to distinguish two cases.

*Case  $|P - Q| \geq \frac{1}{2}|Q|$ :* We can apply Lemma 2.7 for  $p \geq 2$  to conclude

$$\begin{aligned} |H(P, Q)| &\leq |A(P) - A(Q)| + |(DA)(Q)||P - Q| \\ &\leq |A(P) - A(Q)| + c|Q|^{p-2}|P - Q| \\ &\leq |A(P) - A(Q)| + c(|Q| + |P - Q|)^{p-2}|P - Q| \\ &\leq c|A(P) - A(Q)| \\ &\leq c|A(P) - A(Q)| \frac{|P - Q|}{|Q| + |P - Q|}. \end{aligned}$$

*Case  $|P - Q| < \frac{1}{2}|Q|$ :* Using  $|(D^2A)(R)| \leq c|R|^{p-3}$ , we obtain with Taylor's formula

$$\begin{aligned} |H(P, Q)| &= \left| \int_0^1 \left( (DA)(Q + t(P - Q)) - (DA)(Q) \right) dt (P - Q) \right| \\ &\leq \int_0^1 t \int_0^1 |(D^2A)(Q - st(P - Q))| ds dt |P - Q|^2 \\ &\leq c \int_0^1 \int_0^1 |Q - st(P - Q)|^{p-3} ds dt |P - Q|^2. \end{aligned}$$

Since  $|P - Q| < \frac{1}{2}|Q|$ , we have  $\frac{1}{2}|Q| \leq |Q - st(P - Q)| \leq |Q| + |P - Q| \leq 2|Q|$  for all  $s, t \in (0, 1)$ . Thus, Lemma 2.7 yields

$$|H(P, Q)| \leq c(|Q| + |P - Q|)^{p-3}|P - Q|^2 \leq c|A(P) - A(Q)| \frac{|P - Q|}{|Q| + |P - Q|}$$

and the proof is complete.  $\square$

We are now prepared to prove the  $A$ -decay in the non-degenerate case.

**Proposition 3.14.** *Let  $\theta \in (0, 1)$ . Then there exist  $\varepsilon = \varepsilon(\theta) > 0$  and  $c_\theta > 0$  such that if  $u$  satisfies the non-degeneracy condition (3.4) on  $B$  with  $\varepsilon_{\text{DG}} \leq \varepsilon$ , then*

$$\begin{aligned} & \left( \int_{\theta B} |A(\nabla u) - \langle A(\nabla u) \rangle_{\theta B}|^{p'} dx \right)^{\frac{1}{p'}} \\ & \leq c_\theta \left( \int_{2B} |A(\nabla u) - \langle A(\nabla u) \rangle_{2B}|^{p'} dx \right)^{\frac{1}{p'}} + c_\theta \left( \int_{2B} |F - \langle F \rangle_{2B}|^{p'} dx \right)^{\frac{1}{p'}}. \end{aligned}$$

*Proof.* According to Lemma 2.10 we have

$$\begin{aligned} \text{I} & := \int_{\theta B} |A(\nabla u) - \langle A(\nabla u) \rangle_{\theta B}|^{p'} dx \\ & \lesssim \int_{\theta B} |A(\nabla u) - \langle A(\nabla u) \rangle_{\theta B}|^{p'} \chi_{\{|\nabla u - \langle \nabla u \rangle_{\theta B}| \geq \sigma |\langle \nabla u \rangle_{\theta B}|\}} dx \\ & \quad + \int_{\theta B} |A(\nabla u) - \langle A(\nabla u) \rangle_{\theta B}|^{p'} \chi_{\{|\nabla u - \langle \nabla u \rangle_{\theta B}| < \sigma |\langle \nabla u \rangle_{\theta B}|\}} dx \\ & =: \text{II} + \text{III}. \end{aligned}$$

If  $\varepsilon_{\text{DG}}$  is sufficiently small (depending on  $\theta$ ), it follows from Lemma 3.9 that  $u$  also satisfies the non-degeneracy condition (3.4) on  $\theta B$ . Therefore, we get by Lemmata 3.11, 3.10, and 2.10 for any  $0 < \tau < \frac{1}{4}$

$$\begin{aligned} \text{II} & \leq \int_{\theta B} |A(\nabla u) - \langle A(\nabla u) \rangle_{\theta B}|^{p'} \chi_{\{|\nabla u - \langle \nabla u \rangle_{\theta B}^A| \geq \frac{\sigma}{8} |\langle \nabla u \rangle_{\theta B}^A|\}} dx \\ & \lesssim \sigma^{-2p} \tau^{2\gamma} \left( \int_{2\theta B} |A(\nabla u) - \langle A(\nabla u) \rangle_{2\theta B}|^{p'} dx + \tau^{-d-2\gamma} \int_{2\theta B} |F - \langle F \rangle_{2\theta B}|^{p'} dx \right) \\ & \lesssim \sigma^{-2p} \tau^{2\gamma} \theta^{-d} \left( \int_{2B} |A(\nabla u) - \langle A(\nabla u) \rangle_{2B}|^{p'} dx + \tau^{-d-2\gamma} \int_{2B} |F - \langle F \rangle_{2B}|^{p'} dx \right). \end{aligned}$$

Later we will take  $\tau = \tau(\sigma, \theta)$  small such that the factor in front of the  $A$ -oscillation on  $2B$  is small. Of course, the price will be a large factor in front of the  $F$ -oscillation.

Let us now estimate III. We apply Lemma 2.7 and use that  $\sigma \leq 1$  and  $p \geq 2$  to obtain

$$\begin{aligned} \text{III} & \lesssim \int_{\theta B} \left[ (|\langle \nabla u \rangle_{\theta B}| + |\nabla u - \langle \nabla u \rangle_{\theta B}|)^{p-2} |\nabla u - \langle \nabla u \rangle_{\theta B}| \right]^{p'} \chi_{\{|\nabla u - \langle \nabla u \rangle_{\theta B}| < \sigma |\langle \nabla u \rangle_{\theta B}|\}} dx \\ & \lesssim |\langle \nabla u \rangle_{\theta B}|^{(p-2)p'} \int_{\theta B} |\nabla u - \langle \nabla u \rangle_{\theta B}|^{p'} dx. \end{aligned}$$

The latter integral can be estimated further in terms of the solution  $z$  of the linearized equation (3.12). Indeed, from the linear decay (3.13) of solutions to linear systems and  $\theta^{p'} < 1 < \theta^{-d}$  it follows

$$\begin{aligned} \int_{\theta B} |\nabla u - \langle \nabla u \rangle_{\theta B}|^{p'} dx & \lesssim \int_{\theta B} |\nabla z - \langle \nabla z \rangle_{\theta B}|^{p'} dx + \int_{\theta B} |\nabla u - \nabla z|^{p'} dx \\ & \lesssim \sup_{x, x' \in \theta B} |\nabla z(x) - \nabla z(x')|^{p'} + \int_{\theta B} |\nabla u - \nabla z|^{p'} dx \\ & \lesssim \theta^{p'} \int_B |\nabla z - \langle \nabla z \rangle_B|^{p'} dx + \theta^{-d} \int_B |\nabla u - \nabla z|^{p'} dx \\ & \lesssim \theta^{p'} \int_B |\nabla u - \langle \nabla u \rangle_B|^{p'} dx + \theta^{-d} \int_B |\nabla u - \nabla z|^{p'} dx. \end{aligned}$$

In addition, Lemma 3.9 and Lemma 3.8 imply

$$|\langle \nabla u \rangle_{\theta B}| \approx |\langle \nabla u \rangle_B| \approx |\langle \nabla u \rangle_B^A|.$$

Since  $p \geq 2$  we can use this estimate to conclude that

$$\begin{aligned} \text{III} &\lesssim \theta^{p'} |\langle \nabla u \rangle_B^A|^{(p-2)p'} \int_B |\nabla u - \langle \nabla u \rangle_B|^{p'} dx + \theta^{-d} |\langle \nabla u \rangle_B^A|^{(p-2)p'} \int_B |\nabla u - \nabla z|^{p'} dx \\ &=: \text{III}_1 + \text{III}_2. \end{aligned}$$

Now we apply Lemmata 2.10 and 2.7 with  $p \geq 2$  to obtain the following bound on  $\text{III}_1$ :

$$\begin{aligned} \text{III}_1 &\lesssim \theta^{p'} |\langle \nabla u \rangle_B^A|^{(p-2)p'} \int_B |\nabla u - \langle \nabla u \rangle_B^A|^{p'} dx \\ &\lesssim \theta^{p'} \int_B |A(\nabla u) - A(\langle \nabla u \rangle_B^A)|^{p'} dx \\ (3.16) \quad &= \theta^{p'} \int_B |A(\nabla u) - \langle A(\nabla u) \rangle_B|^{p'} dx. \end{aligned}$$

Next, for  $\text{III}_2$  we use the linear comparison of Lemma 3.12 and the estimate for  $H$  from Lemma 3.13 to deduce

$$\begin{aligned} \text{III}_2 &\leq \theta^{-d} c \int_B |H(\nabla u, \langle \nabla u \rangle_B^A)|^{p'} dx + \theta^{-d} \int_B |F - \langle F \rangle_B|^{p'} dx \\ &\lesssim \theta^{-d} \int_B |A(\nabla u) - A(\langle \nabla u \rangle_B^A)|^{p'} \left( \frac{|\nabla u - \langle \nabla u \rangle_B^A|}{|\langle \nabla u \rangle_B^A| + |\nabla u - \langle \nabla u \rangle_B^A|} \right)^{p'} dx + \theta^{-d} \int_B |F - \langle F \rangle_B|^{p'} dx \\ &=: \text{III}_2^A + \text{III}_2^F, \end{aligned}$$

where for  $\sigma, \tau < 1$

$$\text{III}_2^F \lesssim \sigma^{-2p} \tau^{-d} \theta^{-d} \int_{2B} |F - \langle F \rangle_{2B}|^{p'} dx.$$

Note that on the set  $\{x \mid |\nabla u - \langle \nabla u \rangle_B^A| < \sigma |\langle \nabla u \rangle_B^A|\}$  the fraction in the integral of  $\text{III}_2^A$  is smaller than  $\sigma$ , while on its complement it is bounded by one. Hence,

$$\begin{aligned} \text{III}_2^A &\leq \theta^{-d} \int_B |A(\nabla u) - \langle A(\nabla u) \rangle_B|^{p'} \chi_{\{|\nabla u - \langle \nabla u \rangle_B^A| \geq \sigma |\langle \nabla u \rangle_B^A|\}} dx \\ &\quad + \sigma^{p'} \theta^{-d} \int_B |A(\nabla u) - \langle A(\nabla u) \rangle_B|^{p'} \chi_{\{|\nabla u - \langle \nabla u \rangle_B^A| < \sigma |\langle \nabla u \rangle_B^A|\}} dx, \end{aligned}$$

where we have also used  $\langle A(\nabla u) \rangle_B = A(\langle \nabla u \rangle_B^A)$ . Clearly, the second integral is small for small  $\sigma > 0$ . Moreover, the first one can be estimated further by Lemma 3.10 provided that  $\varepsilon = \varepsilon(\sigma, \tau) \geq \varepsilon_{\text{DG}}$  is small enough. Overall, in combination with (3.16) we arrive at

$$\begin{aligned} \text{III} &\lesssim \text{III}_1 + \text{III}_2^A + \text{III}_2^F \\ &\lesssim \theta^{p'} \int_B |A(\nabla u) - \langle A(\nabla u) \rangle_B|^{p'} dx \\ &\quad + \sigma^{-2p} \tau^{2\gamma} \theta^{-d} \left( \int_{2B} |A(\nabla u) - \langle A(\nabla u) \rangle_{2B}|^{p'} dx + \tau^{-d-2\gamma} \int_{2B} |F - \langle F \rangle_{2B}|^{p'} dx \right) \\ &\quad + \sigma^{p'} \theta^{-d} \int_B |A(\nabla u) - \langle A(\nabla u) \rangle_B|^{p'} dx. \end{aligned}$$

Recall that also

$$\int_B |A(\nabla u) - \langle A(\nabla u) \rangle_B|^{p'} dx \leq c \int_{2B} |A(\nabla u) - \langle A(\nabla u) \rangle_{2B}|^{p'} dx.$$



Thus, combing the estimates for II and III we obtain our final estimate

$$\begin{aligned} I &\lesssim \left( \theta^{p'} + \sigma^{-2p} \tau^{2\gamma} \theta^{-d} + \sigma^{p'} \theta^{-d} \right) \int_{2B} |A(\nabla u) - \langle A(\nabla u) \rangle_{2B}|^{p'} dx \\ &\quad + \sigma^{-2p} \tau^{-d} \theta^{-d} \int_{2B} |F - \langle F \rangle_{2B}|^{p'} dx. \end{aligned}$$

Now, we choose the parameters in the following order: Given  $\theta \in (0, 1)$ , we first choose  $\sigma = \sigma(\theta) > 0$  small enough such that  $\sigma^{p'} \theta^{-d} \leq \theta^{p'}$ . Second, we choose  $\tau = \tau(\theta, \sigma, \gamma) = \tau(\theta) > 0$  small such that also  $\sigma^{-2p} \tau^{2\gamma} \theta^{-d} \leq \theta^{p'}$ . Moreover, for the validity of the above estimates (applicability of Lemma 3.10) we have to choose  $\varepsilon = \varepsilon(\sigma, \tau) = \varepsilon(\theta) > 0$  small. Taking the  $p'$ -root then proves our desired  $A$ -decay.  $\square$

**3.4. Proof of Theorems 3.1 and 3.3.** We will now combine the estimates for the degenerate and the non-degenerate case to prove the main results of this section, namely Theorems 3.1 and 3.3.

*Proof of Theorem 3.1.* Let  $\beta \in (0, 1)$ . Define  $\beta_2 := \frac{1+\beta}{2}$  such that  $\beta < \beta_2 < 1$ . We will combine Propositions 3.7 (with  $\beta_2$ ) and 3.14 to prove our claim. To this end, we first choose  $\theta \in (0, 1)$  so small such that  $\theta_0 := \frac{1}{2}\theta$  satisfies  $c\theta^{\beta_2} + c\theta \leq \theta_0^\beta$ , where  $c\theta^{\beta_2}$  is from Proposition 3.7 and  $c\theta$  from Proposition 3.14. In particular this determines  $\varepsilon = \varepsilon(\theta)$  in Proposition 3.14.

If  $u$  satisfies the non-degeneracy condition (3.4) on  $\frac{1}{2}B$ , then the claim follows by Proposition 3.14. If, however, (3.4) is not satisfied on  $\frac{1}{2}B$ , then we can apply Proposition 3.7 to deduce our claim.  $\square$

*Proof of Theorem 3.3.* It follows by repeated use of Theorem 3.1 that for our fixed  $\theta_0$  and all  $k \in \mathbb{N}$  there holds

$$(3.17) \quad \text{osc}_{p'} A(\nabla u)(x, \theta_0^k t) \leq \theta_0^{k\beta} \text{osc}_{p'} A(\nabla u)(x, t) + c \sum_{j=0}^{k-1} \theta_0^{(k-1-j)\beta} \text{osc}_{p'} F(x, \theta_0^j t).$$

Now, let  $\theta \in (0, 1)$ . Then we find  $k \in \mathbb{N}_0$  such that  $\theta_0^{k+1} \leq \theta \leq \theta_0^k$ . Moreover, note that for general  $G \in L^{p'}$ , any  $\lambda \in [\theta_0, 1]$ , and  $s > 0$  we have

$$(3.18) \quad \text{osc}_{p'} G(x, \theta_0 s) \lesssim \text{osc}_{p'} G(x, \lambda s) \lesssim \text{osc}_{p'} G(x, s)$$

with constants only depending on the fixed  $\theta_0$ . Thus, the claim follows from (3.17) in a standard way by changing the discrete sum by an integral using (3.18). This step also introduces the constant in front of  $\theta^\beta$ .  $\square$

**3.5. Consequences and remarks.** In this section we present a few consequences of Theorem 3.1. Let us begin with how our estimates improve the results in [6], where pointwise regularity estimates have been proven for the system version of (1.1) with  $1 < p < \infty$  and  $\Omega \subset \mathbb{R}^d$ . As an important intermediate step they prove an assertion very similar to our Theorem 3.1. For the case  $p \geq 2$  this result reads as follows:

**Proposition 3.15** ([6, Proposition 2.1]). *For  $\Omega \subset \mathbb{R}^d$  and  $p \geq 2$  let  $\gamma$  denote the best possible exponent in Lemma 2.22. If  $u$  solves (1.1), then for  $\beta \in (0, \gamma)$  there exists  $\theta_0 \in (0, 1)$  and  $c_\beta > 0$  such that<sup>1</sup>*

$$\begin{aligned} &\left( \int_{\theta_0 B} |A(\nabla u) - \langle A(\nabla u) \rangle_{\theta_0 B}|^{p'} dx \right)^{\frac{1}{p'}} \\ &\leq \theta_0^\beta \left( \int_B |A(\nabla u) - \langle A(\nabla u) \rangle_B|^{p'} dx \right)^{\frac{1}{p'}} + c_\beta \left( \int_B |F - \langle F \rangle_B|^{p'} dx \right)^{\frac{1}{p'}}. \end{aligned}$$

In contrast to this, our Theorem 3.1 improves the condition on  $\beta$  from  $\beta \in (0, \gamma)$  to  $\beta \in (0, 1)$ . However, our result is restricted to the scalar problem in the plane.

Note that the best value of  $\gamma > 0$  is not known for systems or higher dimensions. Even for the scalar case in the plane it remains open whether the assertion extends to the limiting case  $\beta = 1$ .

<sup>1</sup>It is stated in [6, Proposition 2.1] that  $\beta \in (0, \min\{1, \gamma \frac{2}{p'}\})$ , but this is a typo. It should be  $\beta \in (0, \min\{1, \gamma \frac{2}{\bar{p}}\})$  with  $\bar{p} = \min\{p, 2\}$ .

See, in particular, the discussion in Subsection 2.7. Therefore, it was not possible for us to directly apply Proposition 3.15 but rather make use of the improved  $A$ -decay of Theorem 2.2.

Let us remark that all the results of Subsection 3 are only based on the  $A$ -decay of Theorem 2.2. In particular, if the  $p$ -harmonic system (2.1) is proven to satisfy an  $A$ -decay with power  $\beta$ , then Theorem 3.1 remains valid for all (!) smaller exponents. Hence, all the consequences below will remain valid. Especially all proofs of Subsection 3 are valid for any dimension as well as for systems.

At this point it is also worth to mention that for  $p \geq 2$  a  $V$ -decay with exponent  $\gamma$  directly implies an  $A$ -decay for any exponent  $\beta \in (0, \gamma)$ . This was proven in [15, Remark 5.6]. Hence, our method is more flexible, since we only need  $A$ -decay.

Let us now present some results which improve Theorem 1.3 and Corollary 5.2 of [6]. Particularly, we extend the range of admissible  $\beta$ 's to  $\beta \in (0, 1)$ .

**Corollary 3.16** (General oscillation estimate). *Let  $\Omega \subset \mathbb{R}^2$  and  $p \geq 2$ . Assume that  $u: \Omega \rightarrow \mathbb{R}$  satisfies (1.1). If for  $R > 0$ , some  $\beta \in (0, 1)$ , and  $\omega: (0, R) \rightarrow (0, \infty)$  the function  $r \mapsto \omega(r)r^{-\beta}$  is almost decreasing<sup>2</sup> in  $(0, R)$ , then for any ball  $B_R$  with  $2B_R \subset \Omega$  there holds*

$$M_{\omega, R}^{\sharp, 1}(A(\nabla u))(x) \leq c_\beta M_{\omega, R}^{\sharp, p'}(F)(x) + c_\beta \frac{1}{\omega(R)} \left( \int_{2B_R} |A(\nabla u) - \langle A(\nabla u) \rangle_{2B_R}|^{p'} dx \right)^{\frac{1}{p'}}.$$

Here for any  $q \geq 1$  the localized fractional sharp maximal operator  $M_{\omega, R}^{\sharp, q}$  of  $f \in L_{\text{loc}}^q(\mathbb{R}^2)$  is defined pointwise by

$$M_{\omega, R}^{\sharp, q}(f)(x) := \sup_{\substack{B_r \ni x \\ r < R}} \frac{1}{\omega(r)} \left( \int_{B_r} |f - \langle f \rangle_{B_r}|^q dy \right)^{\frac{1}{q}}.$$

**Corollary 3.17.** *For  $\Omega \subset \mathbb{R}^2$  and  $p \geq 2$  let  $u: \Omega \rightarrow \mathbb{R}$  satisfy (1.1). If  $s \in (0, 1)$  and  $2B \subset \Omega$ , then*

$$|A(\nabla u)|_{C^s(B)} \leq c_s \left( |F|_{C^s(2B)} + \int_{2B} |A(\nabla u) - \langle A(\nabla u) \rangle_{2B}|^{p'} dx \right)^{\frac{1}{p'}}.$$

The proofs of these results are only based on Proposition 3.15, resp. [6, Proposition 2.1]. Our improved version in Theorem 3.1 immediately implies the claims.

Note that Corollary 3.17 is also covered by Theorem 4.1 below since  $C^s = \mathbf{B}_{\infty, \infty}^s$ .

#### 4. REGULARITY TRANSFER - NONLINEAR CALDERÓN-ZYGMUND ESTIMATES

In this section we show that Sobolev regularity up to order one transfers from the right-hand side  $F$  to the flux  $A(\nabla u)$ . We present this result in more general scales of Besov and Triebel-Lizorkin spaces in Subsection 4.1. At first, in Theorem 4.1 it is shown that in terms of quasi-semi norms we have

$$|A(\nabla u)|_{\mathbf{X}(B)} \leq C |F|_{\mathbf{X}(2B)} + \text{lower order terms of } A(\nabla u),$$

where  $\mathbf{X}$  stands for either  $\mathbf{B}_{\varrho, q}^s$  or  $\mathbf{F}_{\varrho, q}^s$ . Afterwards, in Subsection 4.2, we study how this new regularity for  $A(\nabla u)$  translates into regularity statements for  $\nabla u$  and  $V(\nabla u)$ .

**4.1. Regularity transfer from  $F$  to  $A(\nabla u)$ .** For a ball  $B$  let us denote by  $\mathbf{B}_{\varrho, q}^s(B)$  and  $\mathbf{F}_{\varrho, q}^s(B)$ , the Besov space, resp. Triebel-Lizorkin space, of functions on  $B$  with differentiability  $s > 0$ , integrability  $0 < \varrho \leq \infty$ , and fine index  $0 < q \leq \infty$  (with  $\varrho < \infty$  for the  $\mathbf{F}$ -scale). We use  $\|\cdot\|_{\mathbf{B}_{\varrho, q}^s(B)}$  to denote the (quasi-) norm and  $|\cdot|_{\mathbf{B}_{\varrho, q}^s(B)}$  for the (quasi-) semi norm describing the part of the  $s$ -order derivatives. Likewise we do for the  $\mathbf{F}$ -scale. The exact definitions are introduced below. As usual, we let  $(x)_+ := \max\{0, x\}$  for  $x \in \mathbb{R}$ . Then our main result is the following local regularity transfer:

<sup>2</sup>That is, we assume that  $\omega(r)r^{-\beta} \leq c\omega(\varrho)\varrho^{-\beta}$  for all  $0 < r < \varrho < R$ .

**Theorem 4.1.** *Given  $2 \leq p < \infty$ , some domain  $\Omega \subset \mathbb{R}^2$ , and  $F \in L^{p'}(\Omega)$  let  $u \in W^{1,p}(\Omega)$  be a (scalar) weak solution to*

$$-\operatorname{div}(A(\nabla u)) = -\operatorname{div} F \quad \text{in } \Omega.$$

Further, let  $s > 0$  and  $\varrho, q \in (0, \infty]$  be such that

$$(4.1) \quad d\left(\frac{1}{\varrho} - \frac{1}{p'}\right)_+ < s < 1,$$

i.e.,  $\mathbf{B}_{\varrho,q}^s(B) \hookrightarrow L^{p'}(B)$ . Then for any ball  $B$  with  $2B \subset \Omega$  there holds

$$(4.2) \quad |A(\nabla u)|_{\mathbf{B}_{\varrho,q}^s(B)} \lesssim |F|_{\mathbf{B}_{\varrho,q}^s(2B)} + \left( \int_{2B} |A(\nabla u) - \langle A(\nabla u) \rangle_{2B}|^{p'} dx \right)^{\frac{1}{p'}}.$$

If additionally  $\varrho < \infty$  and

$$(4.3) \quad d\left(\frac{1}{q} - \frac{1}{p'}\right)_+ < s < 1,$$

then the same estimate (4.2) holds true when  $\mathbf{B}_{\varrho,q}^s$  is replaced by  $\mathbf{F}_{\varrho,q}^s$ .

**Remark 4.2.** *Our result also generalizes to higher dimensions and vectorial solutions. However, in this setting the differentiability is restricted to  $s \in (0, \beta_0)$ , where  $\beta_0 \leq 1$  is some unknown small number. The reason behind this is the worse decay estimate for  $p$ -harmonic functions in this general situation. In fact, our regularity transfer holds exactly for the same range as the decay estimates. See the also the Remarks 2.4 and 3.2 for more details.*

Let us introduce the norms used to describe our spaces  $\mathbf{B}_{\varrho,q}^s(B)$  and  $\mathbf{F}_{\varrho,q}^s(B)$ , respectively. In order to have the constants in (4.2) independent of the chosen ball  $B = B_R$ , we are using norms that are invariant with respect to the scaling  $g \mapsto g_R := g(\cdot/R)$ . For a ball  $B$  we introduce the scaling invariant  $L^\varrho(B)$  (quasi-) norm by

$$\| \| f \| \|_{L^\varrho(B)} := \begin{cases} \left( \int_B |f|^\varrho dx \right)^{\frac{1}{\varrho}} & \text{for } 0 < \varrho < \infty, \\ \| g \|_{L^\infty(B)} & \text{if } \varrho = \infty. \end{cases}$$

Moreover, we need a localized version of the oscillations from Section 3. For every  $g \in L^w(B)$  with  $1 \leq \omega < \infty$  we define its localized oscillation by

$$\operatorname{osc}_w^B g(x, t) := \left( \int_{B_t(x) \cap B} |g - \langle g \rangle_{B_t(x) \cap B}|^w dy \right)^{\frac{1}{w}}.$$

Using these ingredients there holds the following characterization of  $\mathbf{B}_{\varrho,q}^s(B)$  and  $\mathbf{F}_{\varrho,q}^s(B)$ , resp., which for simplicity we could take here also as a definition.

**Lemma 4.3** (Characterization by oscillations). *Let  $B \subset \mathbb{R}^d$  be a ball with radius  $R$ . Further let  $0 < \varrho, q \leq \infty$ , and  $s > 0$ , as well as  $1 \leq w \leq \infty$ , and assume that*

$$d\left(\frac{1}{\varrho} - \frac{1}{w}\right)_+ < s < 1,$$

i.e.,  $\mathbf{B}_{\varrho,q}^s(B) \hookrightarrow L^w(B)$ , resp.  $\mathbf{F}_{\varrho,q}^s(B) \hookrightarrow L^w(B)$ .

(a) Then there holds

$$\mathbf{B}_{\varrho,q}^s(B) = \{g \in L^{\max\{\varrho, w\}}(B) \mid \|g\|_{\mathbf{B}_{\varrho,q}^s(B)} < \infty\},$$

with  $\|g\|_{\mathbf{B}_{\varrho,q}^s(B)} := \| \| g \| \|_{L^\varrho(B)} + |g|_{\mathbf{B}_{\varrho,q}^s(B)}$ , where

$$|g|_{\mathbf{B}_{\varrho,q}^s(B)} := R^s \left( \int_0^R \left( \frac{\| \operatorname{osc}_w^B g(\cdot, t) \| \|_{L^\varrho(B)}}{t^s} \right)^q dt \right)^{\frac{1}{q}}$$

with the usual modification for  $q = \infty$ .

(b) Additionally, assume  $\varrho < \infty$  and

$$d\left(\frac{1}{q} - \frac{1}{w}\right)_+ < s,$$

Then there holds

$$\mathbf{F}_{\varrho,q}^s(B) = \{g \in L^{\max\{\varrho,w\}}(B) \mid \|g\|_{\mathbf{F}_{\varrho,q}^s(B)} < \infty\},$$

with  $\|g\|_{\mathbf{F}_{\varrho,q}^s(B)} := \|g\|_{L^\varrho(B)} + |g|_{\mathbf{F}_{\varrho,q}^s(B)}$ , where

$$|g|_{\mathbf{F}_{\varrho,q}^s(B)} := R^s \left\| \left( \int_0^R \left( \frac{\text{osc}_w^B g(\cdot, t)}{t^s} \right)^q dt \right)^{\frac{1}{q}} \right\|_{L^\varrho(B)}$$

(modification for  $q = \infty$ ).

*Proof.* For scalar functions this characterization is a special case of Triebel [26, Theorem 2.2.2 and (2.22)] for bounded  $C^\infty$  domains. Our scaling invariant version can be obtained by applying the result of Triebel to the unit ball  $B_1(0)$  and then scale by  $g_R(x) := g(x/R)$  and translate. The extension for vector-valued functions is straightforward.  $\square$

Note that the stated (quasi-) semi norm implicitly depends on the parameter  $w$ . This however only gives rise to equivalent (quasi-) norms in the same space. Moreover, let us recall that the Besov and Triebel-Lizorkin scales of smoothness  $s > 0$  include (among others) the more familiar Hölder-Zygmund spaces  $C^s = \mathbf{B}_{\infty,\infty}^s$ , Sobolev-Slobodeckij spaces  $W^{s,\varrho} = \mathbf{B}_{\varrho,\varrho}^s = \mathbf{F}_{\varrho,\varrho}^s$  (with  $1 \leq \varrho < \infty$  and  $s \notin \mathbb{N}$ ), and Bessel-potential spaces  $H^{s,\varrho} = \mathbf{F}_{\varrho,2}^s$  (with  $1 < \varrho < \infty$ ). For details we refer to [25].

Let us now prove our main result of this Section 4.

*Proof of Theorem 4.1.* For simplicity of presentation we first prove the result in the situation of Banach spaces. That is, for now we assume  $\varrho, q \geq 1$ . The modifications needed for the quasi-Banach case, where the used quasi-triangle inequalities produce additional constants, are explained afterwards.

Let us choose  $\beta \in (s, 1)$ . Then according to Theorem 3.1 we find  $\theta_0 < 1$  and some constant  $c_\beta$  such that the decay estimate

$$(4.4) \quad \text{osc}_{p'} A(\nabla u)(x, \theta_0 t) \leq \theta_0^\beta \text{osc}_{p'} A(\nabla u)(x, t) + c_\beta \text{osc}_{p'} F(x, t)$$

holds for all  $x \in B$  and  $t > 0$  such that  $B_t(x) \subset \Omega$ .

Now, assume that  $q < \infty$ . We start with the representation from Lemma 4.3 (using  $w = p'$ )

$$|A(\nabla u)|_{\mathbf{B}_{\varrho,q}^s(B)} \lesssim R^s \left( \int_0^R \left( \frac{\| \text{osc}_{p'}^B A(\nabla u)(\cdot, t) \|_{L^\varrho(B)}}{t^s} \right)^q dt \right)^{\frac{1}{q}}.$$

For every  $x \in B$  and  $t \in (0, R)$ , we have  $|B_t(x) \cap B| \approx |B_t(x)|$  and therefore we may write  $\text{osc}_{p'}^B A(\nabla u)(\cdot, t) \lesssim \text{osc}_{p'} A(\nabla u)(\cdot, t)$ . This implies

$$(4.5) \quad \begin{aligned} |A(\nabla u)|_{\mathbf{B}_{\varrho,q}^s(B)} &\lesssim R^s \left( \int_0^R \left( \frac{\| \text{osc}_{p'} A(\nabla u)(\cdot, t) \|_{L^\varrho(B)}}{t^s} \right)^q dt \right)^{\frac{1}{q}} \\ &\lesssim R^s \left( \int_0^{\theta_0 R} \left( \frac{\| \text{osc}_{p'} A(\nabla u)(\cdot, t) \|_{L^\varrho(B)}}{t^s} \right)^q dt \right)^{\frac{1}{q}} \\ &\quad + R^s \left( \int_{\theta_0 R}^R \left( \frac{\| \text{osc}_{p'} A(\nabla u)(\cdot, t) \|_{L^\varrho(B)}}{t^s} \right)^q dt \right)^{\frac{1}{q}} \\ &=: \text{I} + \text{II}. \end{aligned}$$

Now, rescaling of the integral and our decay estimate (4.4) imply

$$\begin{aligned} \text{I} &= R^s \left( \int_0^R \left( \frac{\|\text{osc}_{p'} A(\nabla u)(\cdot, \theta_0 t)\|_{L^\varrho(B)}}{(\theta_0 t)^s} \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} \\ &\leq \theta_0^{\beta-s} R^s \left( \int_0^R \left( \frac{\|\text{osc}_{p'} A(\nabla u)(\cdot, t)\|_{L^\varrho(B)}}{t^s} \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} \\ &\quad + c_\beta \theta_0^{-s} R^s \left( \int_0^R \left( \frac{\|\text{osc}_{p'} F(\cdot, t)\|_{L^\varrho(B)}}{t^s} \right)^q \frac{dt}{t} \right)^{\frac{1}{q}}. \end{aligned}$$

Since  $\theta_0^{\beta-s} < 1$  we can absorb the  $\int_0^{\theta_0 R} \cdot dt$ -part of the first integral into I to obtain

$$\begin{aligned} \text{I} &\lesssim R^s \left( \int_{\theta_0 R}^R \left( \frac{\|\text{osc}_{p'} A(\nabla u)(\cdot, t)\|_{L^\varrho(B)}}{t^s} \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} + R^s \left( \int_0^R \left( \frac{\|\text{osc}_{p'} F(\cdot, t)\|_{L^\varrho(B)}}{t^s} \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} \\ &= \text{II} + R^s \left( \int_0^R \left( \frac{\|\text{osc}_{p'} F(\cdot, t)\|_{L^\varrho(B)}}{t^s} \right)^q \frac{dt}{t} \right)^{\frac{1}{q}}. \end{aligned}$$

For  $x \in B$  and  $t \in (0, R)$  we have  $\text{osc}_{p'} F(x, t) = \text{osc}_{p'}^{2B} F(x, t)$ . Thus,

$$\text{I} \lesssim \text{II} + R^s \left( \int_0^R \left( \frac{\|\text{osc}_{p'}^{2B} F(\cdot, t)\|_{L^\varrho(2B)}}{t^s} \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} = \text{II} + |F|_{\mathbf{B}_{\varrho, q}^s(2B)}.$$

Combining this with (4.5) we obtain

$$(4.6) \quad |A(\nabla u)|_{\mathbf{B}_{\varrho, q}^s(B)} \lesssim |F|_{\mathbf{B}_{\varrho, q}^s(2B)} + \text{II}.$$

Moreover, for  $x \in B$  and  $t \in (\theta_0 R, R)$  we have  $B_t(x) \subset 2B \subset \Omega$  and  $|B_t(x)| \approx |2B|$  such that (using Lemma 2.10)

$$(4.7) \quad \text{osc}_{p'} A(\nabla u)(x, t) \lesssim \left( \int_{2B} |A(\nabla u) - \langle A(\nabla u) \rangle_{2B}|^{p'} dy \right)^{\frac{1}{p'}}.$$

Hence,

$$\begin{aligned} \text{II} &\lesssim R^s \left( \int_{\theta_0 R}^R \left( \frac{\|\mathbb{1}\|_{L^\varrho(B)}}{t^s} \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} \left( \int_{2B} |A(\nabla u) - \langle A(\nabla u) \rangle_{2B}|^{p'} dy \right)^{\frac{1}{p'}} \\ &\lesssim \left( \int_{2B} |A(\nabla u) - \langle A(\nabla u) \rangle_{2B}|^{p'} dy \right)^{\frac{1}{p'}} \end{aligned}$$

which together with (4.6) proves the claim for the Besov scale with  $q < \infty$ . The proof for  $q = \infty$  follows by straightforward modifications. Finally, also the statement for the Triebel-Lizorkin scale is shown completely analogously with  $\|\cdot\|_{L^\varrho(B)}$  and  $\int \cdot \frac{dt}{t}$  changing places.

So far we have shown our claim in the case of Banach spaces, i.e., if  $\varrho, q \geq 1$ . Let us now explain the changes for the general quasi-Banach regime  $0 < \min\{\varrho, q\} < 1$ . In this case additional constants might appear in the application of the (quasi-) triangle inequalities for the (quasi-) norms  $\|\cdot\|_\varrho$  and  $(\int |\cdot|^q \frac{dt}{t})^{\frac{1}{q}}$ . Hence,  $\theta_0^{\beta-s}$  has to be replaced by  $c_{\varrho, q} \theta_0^{\beta-s}$ . Thus, our proof still works if  $\theta_0$  is so small that  $c_{\varrho, q} \theta_0^{\beta-s} < 1$ . Unfortunately, this is not guaranteed by Theorem 3.1. However, it immediately follows from Theorem 3.3 with the help of (3.18) that for every fixed  $\theta_1 \in (0, 1)$  we have

$$(4.8) \quad \text{osc}_{p'} A(\nabla u)(x, \theta_1 t) \leq c \theta_1^\beta \text{osc}_{p'} A(\nabla u)(x, t) + c_{\beta, \theta_1} \text{osc}_{p'} F(x, t).$$

So, overall we obtain the factor  $c c_{\varrho, q} \theta_1^{\beta-s}$  instead of  $\theta_0^{\beta-s}$ . For small  $\theta_1$  we can still absorb the terms as in the Banach case. The price to pay is a larger factor in front of the  $F$  terms. Anyhow, this proves the general case.  $\square$

**4.2. Transfer to  $\nabla u$  and  $V(\nabla u)$ .** In this section we show how to transfer the regularity statements for  $A(\nabla u)$  to  $\nabla u$  and  $V(\nabla u)$ . To this end, for fixed  $\alpha > 0$  let us define a transformation  $T_\alpha$  of arbitrary vectors or matrices  $Q$  by

$$Q \mapsto T_\alpha(Q) := |Q|^\alpha \frac{Q}{|Q|}.$$

Then under composition  $\{T_\alpha \mid \alpha > 0\}$  forms a group (with identity  $T_1$  and inverse  $T_\alpha^{-1} = T_{\frac{1}{\alpha}}$ ,  $\alpha > 0$ ). In particular, for  $\alpha, \beta > 0$  there holds  $T_{\alpha\beta}(Q) = T_\alpha(T_\beta(Q))$  and hence

$$(4.9) \quad \nabla u = T_{\frac{1}{p}}(V(\nabla u)) = T_{\frac{1}{p-1}}(A(\nabla u)), \quad \text{as well as} \quad V(\nabla u) = T_{\frac{p'}{2}}(A(\nabla u)).$$

In our situation ( $p \geq 2$ ), we have  $\frac{2}{p}, \frac{1}{p-1}, \frac{p'}{2} \in (0, 1]$ . Therefore, the subsequent proposition is of fundamental importance to us.

**Proposition 4.4.** *Let  $B \subseteq \mathbb{R}^d$  denote some ball and assume  $\alpha \in (0, 1]$ .*

(a) *If  $0 < r \leq \infty$ , then we have*

$$\|T_\alpha(G)\|_{L^{r/\alpha}(B)} = \|G\|_{L^r(B)}^\alpha.$$

(b) *If  $s, \varrho, q$ , and  $w$  satisfy the conditions of Lemma 4.3, then*

$$|T_\alpha(G)|_{\mathbf{B}_{\varrho/\alpha, q/\alpha}^{s, \alpha}(B)} \lesssim |G|_{\mathbf{B}_{\varrho, q}^s(B)}^\alpha.$$

*Moreover, the same is true when the  $\mathbf{B}$  spaces are replaced by  $\mathbf{F}$  spaces.*

We note in passing that Proposition 4.4(a) also holds for the respective scaling invariant norms. Thus, Proposition 4.4(b) can be used to show that  $G \in \mathbf{B}_{\varrho, q}^{s, \alpha}(B)$  implies  $T_\alpha(G) \in \mathbf{B}_{\varrho/\alpha, q/\alpha}^{s, \alpha}(B)$  and likewise for the  $\mathbf{F}$ -case. In fact, also the stated bounds remain true if the (quasi-) semi norms are replaced by the corresponding full (quasi-) norms. A similar statement for the scalar case is contained in [25, Section 5.4]. However, the vectorial setting is different. Therefore, below we will present a general but quite simple proof based on the representation in Lemma 4.3. Before we get to this proof, we need an auxiliary lemma on oscillations.

**Lemma 4.5.** *Let  $\alpha \in (0, 1]$  and  $1 \leq w < \infty$ . Then for all balls  $B$  and  $G \in L^w(B)$ , there holds*

$$\left( \int_B |T_\alpha(G) - \langle T_\alpha(G) \rangle_B|^w dx \right)^{\frac{1}{w}} \lesssim \left( \int_B |G - \langle G \rangle_B|^{\alpha w} dx \right)^{\frac{1}{w}}.$$

*Proof.* Recall that  $T_\alpha(Q) = |Q|^\alpha \frac{Q}{|Q|}$ . Thus  $T_\alpha(Q) = A(Q)$  if we redefine our exponent  $p$  just for this proof (!) as  $p := \alpha + 1 \in (1, 2]$ . Then it follows from Lemma 2.7 (with this  $p$ ) that for all  $P, Q$

$$(4.10) \quad \begin{aligned} |T_\alpha(P) - T_\alpha(Q)| &= |A(P) - A(Q)| \\ &\approx \varphi'_{|Q|}(|P - Q|) \\ &\approx (|Q| + |P - Q|)^{\alpha-1} |P - Q| \\ &\leq |P - Q|^\alpha. \end{aligned}$$

Using Lemma 2.10 and the bijectivity of  $T_\alpha$ , and (4.10) we estimate

$$\begin{aligned} \left( \int_B |T_\alpha(G) - \langle T_\alpha(G) \rangle_B|^w dx \right)^{\frac{1}{w}} &\approx \inf_{H_0} \left( \int_B |T_\alpha(G) - H_0|^w dx \right)^{\frac{1}{w}} \\ &= \inf_{G_0} \left( \int_B |T_\alpha(G) - T_\alpha(G_0)|^w dx \right)^{\frac{1}{w}} \\ &\lesssim \inf_{G_0} \left( \int_B |G - G_0|^{\alpha w} dx \right)^{\frac{1}{w}} \\ &\approx \left( \int_B |G - \langle G \rangle_B|^{\alpha w} dx \right)^{\frac{1}{w}}, \end{aligned}$$

where the infimum is taken over all constants  $G_0$ , resp.  $H_0$ . □

**Remark 4.6.** Note that in Lemma 4.5 it is possible to replace  $B$  by  $B \cap B_t(x)$  for each  $x \in B$  and all  $t \in (0, R]$ , where  $R$  denotes the radius of  $B$ .

We are now prepared to prove Proposition 4.4.

*Proof of Proposition 4.4.* The formula  $\|T_\alpha(G)\|_{L^{r/\alpha}(B)} = \|G\|_{L^r(B)}^\alpha$  is obvious for all  $r \in (0, \infty]$ . So, let us now show part (b), i.e.,  $|T_\alpha(G)|_{\mathbf{B}_{\varrho/\alpha, q/\alpha}^{\alpha s}(B)} \lesssim |G|_{\mathbf{B}_{\varrho, q}^s(B)}$ . For this purpose, we will use the characterization of Lemma 4.3. It follows from Lemma 4.5 and Remark 4.6 that

$$(4.11) \quad \text{osc}_{w/\alpha}^B(T_\alpha(G))(x, t) \lesssim (\text{osc}_w^B G(x, t))^\alpha.$$

Moreover,  $s, \varrho, q, w$  replaced by  $\alpha s, \varrho/\alpha, q/\alpha, w/\alpha$  also satisfy the conditions of Lemma 4.3. Thus, we can calculate

$$\begin{aligned} |T_\alpha(G)|_{\mathbf{B}_{\varrho/\alpha, q/\alpha}^{\alpha s}(B)} &= R^s \left( \int_0^R \left( \frac{\|\text{osc}_{w/\alpha}^B T_\alpha(G)(\cdot, t)\|_{L^{\varrho/\alpha}(B)}}{t^{\alpha s}} \right)^\frac{\alpha}{\alpha} \frac{dt}{t} \right)^\frac{\alpha}{\alpha} \\ &\lesssim R^s \left( \int_0^R \left( \frac{\|(\text{osc}_w^B G(\cdot, t))^\alpha\|_{L^{\varrho/\alpha}(B)}}{t^{\alpha s}} \right)^\frac{\alpha}{\alpha} \frac{dt}{t} \right)^\frac{\alpha}{\alpha} \\ &= R^s \left( \int_0^R \left( \frac{\|\text{osc}_w^B G(\cdot, t)\|_{L^{\varrho}(B)}}{t^s} \right)^q \frac{dt}{t} \right)^\frac{\alpha}{\alpha} \\ &= |G|_{\mathbf{B}_{\varrho, q}^s(B)}. \end{aligned}$$

This proves the  $\mathbf{B}_{\varrho, q}^s$ -estimate (with the obvious modifications for  $q = \infty$ ). The  $\mathbf{F}_{\varrho, q}^s$ -case is shown analogously.  $\square$

We can now combine Theorem 4.1 and Proposition 4.4 with the representations (4.9) to conclude new regularity results for  $\nabla u$  and  $V(\nabla u)$ .

**Corollary 4.7.** Under the assumptions of Theorem 4.1 there holds

$$\|\nabla u\|_{L^{\varrho(p-1)}(B)}^{p-1} = \|V(\nabla u)\|_{L^{2\varrho/p'}(B)}^{\frac{2}{p'}} = \|A(\nabla u)\|_{L^{\varrho}(B)}$$

and

$$\begin{aligned} |\nabla u|_{\mathbf{B}_{\varrho(p-1), q(p-1)}^{\frac{p-1}{p}}(B)}^{p-1} &\lesssim |V(\nabla u)|_{\mathbf{B}_{\frac{2\varrho}{p'}, \frac{2q}{p'}}^{\frac{p'}{2}}(B)}^{\frac{2}{p'}} \\ &\lesssim |A(\nabla u)|_{\mathbf{B}_{\varrho, q}^s(B)} \\ &\lesssim |F|_{\mathbf{B}_{\varrho, q}^s(2B)} + \left( \int_{2B} |A(\nabla u) - \langle A(\nabla u) \rangle_{2B}|^{p'} dx \right)^\frac{1}{p'}. \end{aligned}$$

Under the same additional assumptions as in Theorem 4.1 the latter estimates remain true in the scale of Triebel-Lizorkin spaces.

**Remark 4.8.** Let us compare Corollary 4.7 to the results from [8]. They prove for  $p \geq 2, d \geq 2, s \in (0, 1)$ , and  $1 \leq q \leq \frac{2d}{d-2s}$  that locally

$$F \in \mathbf{B}_{2, q}^s \quad \text{implies} \quad V(\nabla u) \in \mathbf{B}_{\frac{2}{p'}, \frac{2q}{p}}^{\frac{p'}{2}}.$$

Our result applied to the same situation ( $\varrho = 2$  in dimension  $d = 2$ ) yields that

$$F \in \mathbf{B}_{2, q}^s \quad \text{implies} \quad V(\nabla u) \in \mathbf{B}_{\frac{4}{p'}, \frac{2q}{p}}^{\frac{p'}{2}}.$$

In particular, the integrability of  $V(\nabla u)$  is increased from 2 to  $\frac{4}{p'}$  and we need no restrictions on the fine index  $q$ .

**Remark 4.9.** Let us compare our results also to the findings in [10]. They have studied the Besov regularity of  $u$  measured in the  $L^p$  adaptivity scale, i.e., those  $\mathbf{B}_{\tau,\tau}^\sigma$  with  $\sigma - \frac{d}{\tau} = -\frac{d}{p}$ . For  $p \geq 2$ , Lipschitz domains in  $d = 2$ , and  $f := -\operatorname{div} F \in L^\infty$  they show that globally  $u \in \mathbf{B}_{\tau,\tau}^\sigma$  for all  $\sigma \in (0, p')$ . However, for  $f \in L^\varrho$  with  $\varrho \in (2p, \infty)$  they obtain the condition  $\sigma < 1 + \frac{1-2/\varrho}{p-1}$ . So their upper bound for  $\sigma$  depends on  $\varrho$ . The reason for this is that the lower integrability of  $f$  induces a smaller exponent  $\alpha_\varrho^* = \frac{1-2/\varrho}{p-1}$  of local Hölder continuity for  $\nabla u$ . Hence, with the techniques from [10] it is only possible to treat differentiability up to  $s < \alpha_\varrho^*$ . Moreover, for  $f \in L^\varrho$  with  $\varrho \in (2, 2p]$  they are restricted to  $\sigma < 1 + \frac{1}{p}$  which is again non optimal. However, our interior oscillations estimates do not require the use of Hölder spaces. Thus, this unnatural bound does not appear. To see this, let us now assume that  $f \in L^\varrho$  with  $\varrho \geq p'$  and  $d = 2$ . Using the Bogovskiĭ operator we find  $F \in \mathbf{F}_{\varrho,2}^1$  with  $\operatorname{div} F = f$ . So  $F \in \mathbf{B}_{p',q}^s$  for any  $s < 1$  and  $q \in (0, \infty]$  such that from Corollary 4.7 it follows that locally in the interior there holds  $u \in \mathbf{B}_{p,q}^{1+\frac{s}{p-1}}$ . Note that independently of  $\varrho$  this yields  $u \in \mathbf{B}_{\tau,\tau}^\sigma$  for all  $\sigma \in (0, p')$  which improves [10].

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