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Abstract

We consider a doubly nonlinear evolution equation with multiplicative noise and show existence and uniqueness of a strong solution. Using a semi-implicit time discretization we get approximate solutions. The theorems of Prokhorov and Skorokhod will give us a.s. convergence in a new probability space, which allows to show the existence of martingale solutions. By pathwise uniqueness we are able to show existence and uniqueness of strong solutions.

1 Introduction

We consider the doubly nonlinear PDE with multiplicative noise:

$$
\begin{aligned}
d(B(u)) - \operatorname{div} A(\nabla u)\ dt &= H(u)\ dW && \text{in } \Omega \times Q_T, \\
u &= 0 && \text{on } \Omega \times (0,T) \times \partial D, \\
u(\cdot, 0) &= u_0 \in W_0^{1,p}(D) && \text{in } \Omega \times D,
\end{aligned}
$$

where $(\Omega, \mathcal{F}, P)$ is a complete, countably generated probability space, $D \subset \mathbb{R}^d$ is a bounded Lipschitz domain, $T > 0$, $Q_T := (0,T) \times D$ and $p > 2$. In the following we will denote this stochastic evolution problem by (P).

We assume that $b : \mathbb{R} \to \mathbb{R}$ is a differentiable function with $b(0) = 0$. For a measurable function $u : D \to \mathbb{R}$ we define $B(u)(x) := b(u(x))$ for almost every $x \in D$.

Moreover, we assume that $a : D \times \mathbb{R}^d \to \mathbb{R}^d$ is a Carathéodory function, i.e., $D \ni x \mapsto a(x,\xi)$ is measurable for all $\xi \in \mathbb{R}^d$ and $\mathbb{R}^d \ni \xi \mapsto a(x,\xi)$ is continuous for almost every $x \in D$. For a measurable function $G : D \to \mathbb{R}^d$ we define $A(G)(x) := a(x, G(x))$ for almost every $x \in D$.

The space of Hilbert-Schmidt operators from $\mathcal{U}$ to $\mathcal{H}$ will be denoted by $HS(\mathcal{U},\mathcal{H})$, where $\mathcal{U}$ and $\mathcal{H}$ are separable Hilbert spaces. Shortly, we set $HS(\mathcal{U}) := HS(\mathcal{U},\mathcal{U})$. Then

H is an operator from $L^2(D)$ to $HS(L^2(D))$.

We define $W(t)$ as a cylindrical Wiener process with values in $L^2(D)$ with respect to a filtration $(\mathcal{F}_t)_{t\in[0,T]}$ satisfying the usual assumptions, i.e., for an orthonormal basis (e_n) of $L^2(D)$ and a sequence of real-valued, independent Brownian motions $(\beta_n)_n$ with respect to a filtration $(\mathcal{F}_t)$ we define

$$W(t) := \sum_{n=1}^{\infty} e_n \beta_n(t).$$

This process can be interpreted as a Q-Wiener process with covariance matrix $Q = \text{diag}(\frac{1}{n^2})$ and values in U, where U is the completion of $L^2(D)$ with respect to the scalar product

$$(u, v)_U := \sum_{n=1}^{\infty} \frac{(v, e_n)_2 (u, e_n)_2}{n^2},$$

$u, v \in L^2(D)$ (see [10], Section 4.1, 4.2).

More precise assumptions on A, H and B are given in the next section.

The techniques used in this contribution are adapted from [14]. More precisely, in [14] we find the situation where $b = Id$ and the monotone operator $-\text{div } A = -\Delta_p$ is perturbed by a strongly continuous, fist-order term $-\text{div } F$ with $F : \mathbb{R} \to \mathbb{R}^d$ Lipschitz continuous. Consequently, the operator under consideration in [14] is pseudomonotone. In our case $F = 0$, therefore monotonicity methods apply to the (more general) diffusion term $-\text{div } a(x, \nabla u)$ but technical difficulties arise from the nonlinear term $b(u)$ in the time derivative. Therefore, some of the arguments from [14] have to be changed and completely new arguments have been added in our setting. However, for $v := b(u)$ the equation in (P) is equivalent to

$$dv - \text{div } a(x, \nabla b^{-1}(v)) \, dt = H(b^{-1}(v)) \, dW$$

and the operator $A : W_0^{1,p}(D) \to W^{-1,p'}(D)$, $Av = -\text{div } a(x, \nabla b^{-1}(v))$ is pseudomonotone (see Appendix, Theorem 7.1), but of a different structure than the one considered in [14]. We should note that the operator $H \circ b^{-1}$ satisfies the property (H1) in section 2 if and only if H satisfies it. The existence and uniqueness of solutions for a stochastic evolution equation with a general pseudomonotone operator is, to the best of our knowledge, an open problem.
In Section 3 we will present our main theorems. These are Theorem 3.4 and Theorem 3.5. In Section 4 one can find the proof of Theorem 3.4. To prove Theorem 3.4 we will first solve the corresponding semi-implicit time discrete problem to get approximate solutions. Since we cannot get any a.s. convergence for the approximate solutions, we use the theorems of Prokhorov and Skorokhod to get a.s. convergence of the approximate solutions v_N to v_∞ with respect to a new probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P})$. Firstly, we show that

a limit equation holds true. Secondly, we have to identify the limit of the approximative stochastic integrals with a stochastic integral, where the integrand is $H(v_\infty)$. Thirdly, we identify the weak limit of $A(\nabla v_N)$ with $A(\nabla v_\infty)$ by using an Itô formula for the limit equation and a Minty type monotonicity argument (see e.g., [11]).
In Section 5 we show pathwise uniqueness of a solution of (P) with respect to the same probability space, the same filtration, the same Wiener process and the same initial value. To do this, we use a generalized version of the Itô formula presented by Pardoux [9] in a Gelfand-triple setting. This will be used to prove Theorem 3.5 in Section 6. There we construct two sequences of approximate solutions which converge to a solution of (P) in the same probability space. Since this solution is unique we get convergence in probability of approximate solutions in the initial probability space $(\Omega, \mathcal{F}, P)$ to a strong solution of (P) which is again unique.

2 Technical Assumptions

2.1 Assumptions on A

We assume that the following assumptions hold true for $a : D \times \mathbb{R}^d \to \mathbb{R}^d$:
(a1) a is monotone with respect to the second component, i.e.,

$$(a(x,\xi) - a(x,\zeta)) \cdot (\xi - \zeta) \geq 0$$

for almost every $x \in D$ and all $\xi, \zeta \in \mathbb{R}^d$.
(a2) a is coercive, i.e., there exists a constant $c_1 > 0$ and $k_1 \in L^1(D)$ such that

$$a(x,\xi) \cdot \xi \geq c_1 |\xi|^p - k_1(x)$$

for almost every $x \in D$ and all $\xi \in \mathbb{R}^d$.
(a3) a is bounded, i.e., there exists a constant $c_2 > 0$ and $k_2 \in L^{p'}(D)$ such that

$$|a(x,\xi)| \leq c_2 |\xi|^{p-1} + k_2(x)$$

for almost every $x \in D$ and all $\xi \in \mathbb{R}^d$.

Remark 2.1.1. *From assumptions i), ii) and iii) it follows that $A : L^p(D)^d \to L^{p'}(D)^d$ defined as $(Au)(x) := a(x, u(x))$ for $u \in L^p(D)^d$ and almost every $x \in D$, satisfies the following properties:*
(A1) A is monotone, i.e.,

$$(Au - Av, u - v)_{(L^{p'}(D)^d, L^p(D)^d)} \geq 0$$

for all $u, v \in L^p(D)^d$.
(A2) A is coercive, i.e.,

$$(Au, u)_{(L^{p'}(D)^d, L^p(D)^d)} \geq c_1 \|u\|^p_{L^p(D)^d} - \|k_1\|_1$$

for all $u \in L^p(D)^d$.
(A3) A is bounded, i.e.,

$$\|Au\|_{L^{p'}(D)^d} \le c_2 \|u\|^{p-1}_{L^p(D)^d} + \|k_2\|_{p'}$$

for all $u \in L^p(D)^d$. *By a standard argument of Nemyckii operators (see e.g. [11], p.72-73) one can see that* $A : L^p(D)^d \to L^{p'}(D)^d$ *is continuous.*

2.2 Assumptions on H

For the orthonormal basis (e_n) of $L^2(D)$ as in Section 1 and $u \in L^2(D)$ we define

$$H(u)(e_n) := h_n \circ u,$$

where, for any $n \in \mathbb{N}$, $h_n : \mathbb{R} \to \mathbb{R}$ is a continuously differentiable function with $h_n(0) = 0$ satisfying the following assumptions:

(H1) There exists a constant $C > 0$ such that

$$\sum_{n=1}^{\infty} \|h_n'\|_\infty^2 \le C.$$

An immediate consequence from (H1) is:
(H2) The sequence $(h_n)_n$ fulfills the inequality

$$\sum_{n=1}^{\infty} |h_n(\lambda) - h_n(\mu)|^2 \le C|\lambda - \mu|^2$$

for all $\lambda, \mu \in \mathbb{R}$.
In particular, for $u \in L^2(D)$ we have

$$\|H(u)\|^2_{HS(L^2(D))} = \sum_{n=1}^{\infty} \|H(u)(e_n)\|_2^2 = \sum_{n=1}^{\infty} \int_D |h_n(u(x))|^2 \, dx \le C\|u\|_2^2.$$

Proposition 2.2.1. $H : W_0^{1,p}(D) \to HS(L^2(D), H_0^1(D))$ *is continuous.*

Proof. See [14], p.83-84. □

Remark 2.2.2. *For any* $u \in W_0^{1,p}(D)$, *by Young inequality, we get*

$$\|H(u)\|^p_{HS(L^2(D);H_0^1(D))} = \left(\sum_{n=1}^{\infty} \|h_n(u)\|^2_{H_0^1(D)}\right)^{\frac{p}{2}}$$
$$\le \left(\sum_{n=1}^{\infty} \|h_n'\|_\infty^2 \int_D |\nabla u|^2 \, dx\right)^{\frac{p}{2}} \le C^{\frac{p}{2}} C_p \|\nabla u\|_p^p.$$

for a constant $C_p > 0$.

2.3 Assumptions on B

We assume that $b:\mathbb{R}\to\mathbb{R}$ satisfies the following assumptions:

(B1) $b':\mathbb{R}\to\mathbb{R}$ is Lipschitz continuous and there exist constants $c,\tilde{c}>0$ such that

$$c\leq b'\leq \tilde{c}.$$

Under these assumptions it is clear that b is strictly monotone and coercive, hence it is bijective. The inverse b^{-1} is differentiable and the derivative satisfies $\frac{1}{\tilde{c}}\leq (b^{-1})'\leq \frac{1}{c}$. For example, these assumptions are fulfilled by functions like $b=Id+\arctan$, $b=2Id+\sin$ or $b=2Id+\cos$.

It is easy to see that $B:L^2(D)\to L^2(D)$, defined as $B(u)(x):=b(u(x))$ for all $u\in L^2(D)$ and almost every $x\in D$, is Lipschitz continuous and strongly monotone. Hence by the theorem of Zarantonello (see [13], p.504, Theorem 25.B) it is bijective with Lipschitz continuous inverse $B^{-1}:L^2(D)\to L^2(D)$. In particular, for $u\in L^2(D)$ we have:

$$\|u\|_2\leq \frac{1}{c}\|B(u)\|_2.$$

For $u\in W_0^{1,p}(D)$, according to the chain rule for Sobolev functions we have $B(u)\in W_0^{1,p}(D)$ and $\|\nabla B(u)\|_p^p=\|b'(u)\nabla u\|_p^p$.

3 Strong and martingale solutions and the main theorems

In the following we define strong and martingale solutions to our problem (P). These definitions of a solution are standard in the theory of stochastic evolution equations.

Definition 3.1 (**Strong solution**). *For an arbitrary $u_0\in L^2(D)$ we call a predictable process ([10], p. 27-28) $u:\Omega\times[0,T]\to L^2(D)$ a* ***strong solution*** *to (P) if and only if*

$$B(u(\omega,\cdot))\in\mathcal{C}([0,T];W^{-1,p'}(D))\cap L^\infty(0,T;L^2(D))$$

for almost every $\omega\in\Omega$, $u\in L^p(\Omega;L^p(0,T;W_0^{1,p}(D)))$, $u(\cdot,0)=u_0$ a.s. in Ω and

$$B(u(t))-B(u_0)-\int_0^t \operatorname{div} A(\nabla u)\,ds=\int_0^t H(u)\,dW$$

in $L^2(D)$, for all $t\in[0,T]$, a.s. in Ω.

Remark 3.2. *We remark that*

$$C([0,T];W^{-1,p'}(D))\cap L^\infty(0,T;L^2(D))\subset\mathcal{C}_w([0,T];L^2(D)),$$

so $B(u(t))\in L^2(D)$ does make sense for all $t\in[0,T]$, a.s. in Ω. Since $u(t)=B^{-1}(B(u(t)))$ for all $t\in[0,T]$, a.s. in Ω, $u(t)\in L^2(D)$ does also make sense for all $t\in[0,T]$, a.s. in Ω and $u:\Omega\times[0,T]\to L^2(D)$ is a stochastic process if and only if $B(u):\Omega\times[0,T]\to L^2(D)$ is a stochastic process.

Often it is necessary to consider the probability space, the filtration and the Wiener process as unknowns of the problem. In particular, this is the case if one wants to use the theorems of Prokhorov and Skorokhod to get a.s. convergence of the approximate solutions. The corresponding definition of a solution of (P) is the following.

Definition 3.3 **(Martingale solution).** *We say (P) has a* ***martingale solution****, if and only if for an arbitrary* $u_0 \in L^2(D)$ *there exists a probability space* $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P})$*, a filtration* $(\hat{\mathcal{F}}_t)_{t\in[0,T]}$ *and a cylindrical Wiener process* $\hat{W}$ *with values in* $L^2(D)$ *such that there exists a predictable process* $u : \hat{\Omega} \times [0,T] \to L^2(D)$ *such that*

$$B(u(\hat{\omega}, \cdot)) \in \mathcal{C}([0,T]; W^{-1,p'}(D)) \cap L^\infty(0,T; L^2(D))$$

for almost every $\hat{\omega} \in \hat{\Omega}$*,* $u \in L^p(\hat{\Omega}; L^p(0,T; W_0^{1,p}(D)))$*,* $u(\cdot, 0) = u_0$ *a.s. in* $\hat{\Omega}$ *and*

$$B(u(t)) - B(u_0) - \int_0^t \text{div } A(\nabla u) \; ds = \int_0^t H(u) \; d\hat{W}$$

in $L^2(D)$*, for all* $t \in [0,T]$*, a.s. in* $\hat{\Omega}$*.*

Our aim is to prove the following two theorems:

Theorem 3.4. *For any* $u_0 \in W_0^{1,p}(D)$ *there exists a martingale solution to (P).*

Theorem 3.5. *For any* $u_0 \in W_0^{1,p}(D)$ *there exists a unique strong solution to (P).*

4 Proof of Theorem 3.4

4.1 Semi-implicit time discretization

For $N \in \mathbb{N}$ and $k = 0, ..., N$ define $\tau := \frac{T}{N}$ and $t_k := k \cdot \tau$. Thus $t_0 = 0 < t_1 < ... < t_N = T$ is an equidistant decomposition of the time interval $[0,T]$.
For $u_0 \in L^2(D)$ we consider the following semi-implicit time discrete problem

$$B(u^{k+1}) - B(u^k) - \tau \text{ div } A(\nabla u^{k+1}) = H(u^k)\Delta_{k+1}W, \tag{1}$$

where $\Delta_{k+1}W := W(t_{k+1}) - W(t_k)$ for $k = 0, ..., N-1$.

Lemma 4.1.1. *For any* $u_0 \in L^2(D)$ *and any* $k = 0, ..., N-1$ *there exist unique* $\mathcal{F}_{t_{k+1}}$*-measurable functions* $u^{k+1} : \Omega \to W_0^{1,p}(D)$ *such that for almost every* $\omega \in \Omega$

$$B(u^{k+1}) - B(u^k) - \tau \text{ div } A(\nabla u^{k+1}) = H(u^k)\Delta_{k+1}W$$

in $L^2(D)$*.*

Proof. By induction we assume the existence and uniqueness of u^k as in the lemma, and we want to show the existence and uniqueness of u^{k+1}. We consider the equivalent equation

$$B(u^{k+1}) - \tau \text{ div } A(\nabla u^{k+1}) = H(u^k)\Delta_{k+1}W + B(u^k)$$

and set $S_\tau : W_0^{1,p}(D) \to W^{-1,p'}(D)$,

$$(S_\tau(u), v)_{(W^{-1,p'},W_0^{1,p})} := (B(u), v)_2 + \tau \int_D A(\nabla u) \cdot \nabla v \, dx.$$

Since A is monotone and B is strongly monotone, S_τ is strictly monotone.
For $u \in W_0^{1,p}(D)$ we have $(B(u), u)_2 \geq 0$, so we conclude

$$(S_\tau(u), u)_{(W^{-1,p'},W_0^{1,p})} \geq \tau \int_D A(\nabla u) \cdot \nabla u \, dx \geq \tau(c_1 \|\nabla u\|_p^p - \|k_1\|_1).$$

Therefore S_τ is coercive. Since $B : L^2(D) \to L^2(D)$ and A are continuous, S_τ is continuous.
Hence, by the theorem of Minty-Browder (see [11], p. 63) S_τ is bijective. It follows that there exists a unique function $u^{k+1} : \Omega \to W_0^{1,p}(D)$ such that the time discrete equation (1) holds true. It is left to show that u^{k+1} is $\mathcal{F}_{t_k+1}$-measurable. Since $W_0^{1,p}(D)$ is separable, by the theorem of Dunford-Pettis (see [11], p. 35) it is sufficient to show that S_τ^{-1} is demi-continuous.
For $f \in W^{-1,p'}(D)$ there exists a unique $u \in W_0^{1,p}(D)$ such that $S_\tau(u) = f$. By (B1) and (A2) we have

$$\begin{aligned} & c\|u\|_2^2 + \tau(c_1\|\nabla u\|_p^p - \|k_1\|_1) \leq (B(u), u)_2 + \tau(A(\nabla u), \nabla u)_{p',p} \\ =& (S_\tau(u), u)_{(W^{-1,p'},W_0^{1,p})} = (f, u)_{(W^{-1,p'},W_0^{1,p})} \leq \|f\|_{W^{-1,p'}(D)} \|u\|_{W_0^{1,p}(D)} \\ \leq& C_\tau \|f\|_{W^{-1,p'}(D)}^{p'} + \frac{c_1\tau}{2}\|\nabla u\|_p^p \end{aligned}$$

for a constant $C_\tau > 0$. It follows

$$c\|u\|_2^2 + \frac{c_1\tau}{2}\|\nabla u\|_p^p \leq C_\tau \|f\|_{W^{-1,p'}(D)}^{p'} + \tau\|k_1\|_1.$$

Let $f_n \to f$ in $W^{-1,p'}(D)$ and set $u_n := S_\tau^{-1}(f_n)$. The calculation above shows that u_n is bounded in $W_0^{1,p}(D)$. Hence there exists a not relabeled subsequence and $u \in W_0^{1,p}(D)$ such that $u_n \rightharpoonup u$ in $W_0^{1,p}(D)$. Since $W_0^{1,p}(D)$ is compactly embedded in $L^p(D)$ we get $u_n \to u$ in $L^p(D)$, in particular in $L^2(D)$. Now by (A3) the sequence $A(\nabla u_n)$ is bounded in $L^{p'}(D)^d$, so there exists a not relabeled subsequence and $G \in L^{p'}(D)^d$ such that $A(\nabla u_n) \rightharpoonup G$ in $L^{p'}(D)$. Now we get

$$\begin{aligned} & (B(u), u)_2 + \limsup_{n\to\infty} \Big(\tau(A(\nabla u_n), \nabla u_n)_{p',p}\Big) = \limsup_{n\to\infty} \Big((B(u_n), u_n)_2 + \tau(A(\nabla u_n), \nabla u_n)_{p',p}\Big) \\ =& \limsup_{n\to\infty} \, (S_\tau(u_n), u_n)_{(W^{-1,p'},W_0^{1,p})} = \limsup_{n\to\infty} \, (f_n, u_n)_{(W^{-1,p'},W_0^{1,p})} \\ =& (f, u)_{(W^{-1,p'},W_0^{1,p})} = \lim_{n\to\infty} (f_n, u)_{(W^{-1,p'},W_0^{1,p})} \\ =& \lim_{n\to\infty} (S_\tau(u_n), u)_{(W^{-1,p'},W_0^{1,p})} = \lim_{n\to\infty} \Big((B(u_n), u)_2 + \tau(A(\nabla u_n), \nabla u)_{p',p}\Big) \\ =& (B(u), u)_2 + \tau(G, \nabla u)_{p',p}, \end{aligned} \tag{2}$$

which leads to

$$\limsup_{n\to\infty} \, (A(\nabla u_n), \nabla u_n)_{p',p} = (G, \nabla u)_{p',p}.$$

Since A is monotone and continuous, it fulfills the (M)-property (see [11], p.74-75). This leads to the equation $G = A(\nabla u)$.
Because of the fact that S_τ is monotone and continuous, S_τ also fulfills the (M)-property. Since

$$\begin{aligned} u_n &\rightharpoonup u \text{ in } W_0^{1,p}(D), \\ S_\tau(u_n) &\to f \text{ in } W^{-1,p'}(D), \\ (f,u)_{(W^{-1,p'},W_0^{1,p})} &= \limsup_{n\to\infty} \, (S_\tau(u_n), u_n)_{(W^{-1,p'},W_0^{1,p})} \end{aligned}$$

we get $S_\tau(u) = f$ and therefore $S_\tau^{-1}(f) = u$. This means

$$S_\tau^{-1}(f_n) = u_n \rightharpoonup u = S_\tau^{-1}(f)$$

in $W_0^{1,p}(D)$. By the subsequence principle this weak convergence holds true for the whole sequence f_n, hence S_τ^{-1} is demi-continuous. □

4.2 A-priori estimates

Lemma 4.2.1. *For $u_0 \in W_0^{1,p}(D)$ and $k = 0, ..., N-1$ let u^{k+1} be a solution to (1). Then, for all $n = 1, ..., N$ we have the inequality*

$$\begin{aligned} &\frac{1}{2}E\|B(u^n)\|_2^2 - \frac{1}{2}\|B(u_0)\|_2^2 + \frac{1}{4}E\sum_{k=0}^{n-1}\|B(u^{k+1}) - B(u^k)\|_2^2 + c_1 c\tau E\sum_{k=0}^{n-1}\|\nabla u^{k+1}\|_p^p \\ \leq &\tilde{c}T\|k_1\|_1 + \frac{C_1}{c^2}\tau E\sum_{k=0}^{n-1}\|B(u^k)\|_2^2. \end{aligned}$$

Proof. We take the L^2-scalar product with $B(u^{k+1})$ in the time discrete equation (1) and get

$$(B(u^{k+1}) - B(u^k), B(u^{k+1}))_2 + \tau\int_D A(\nabla u^{k+1}) \cdot \nabla B(u^{k+1}) \, dx = (H(u^k)\Delta_{k+1}W, B(u^{k+1}))_2. \tag{3}$$

Using the identity $(x-y)x = \frac{1}{2}(x^2 - y^2 + (x-y)^2)$ for all $x, y \in \mathbb{R}$ we may conclude

$$(B(u^{k+1}) - B(u^k), B(u^{k+1}))_2 = \frac{1}{2}\Big(\|B(u^{k+1})\|_2^2 - \|B(u^k)\|_2^2 + \|B(u^{k+1}) - B(u^k)\|_2^2\Big). \tag{4}$$

It is easy to see that by the assumptions (A2) and (B1) we obtain

$$\begin{aligned}
&\tau\int_D A(\nabla u^{k+1})\cdot\nabla B(u^{k+1})\,dx = \tau\int_D A(\nabla u^{k+1})\cdot\nabla u^{k+1}b'(u^{k+1})\,dx\\
=&\tau\int_D (A(\nabla u^{k+1})\cdot\nabla u^{k+1}+k_1(x))b'(u^{k+1})-k_1(x)b'(u^{k+1})\,dx\\
\geq& c\tau\int_D c_1|\nabla u^{k+1}|^p\,dx-\tilde{c}\tau\int_D |k_1(x)|\,dx. \qquad (5)
\end{aligned}$$

As $B(u^k)$ is $\mathcal{F}_{t_k}$-measurable and $\Delta_{k+1}W$ is independent of $\mathcal{F}_{t_k}$ we have

$$E(H(u^k)\Delta_{k+1}W, B(u^k))_2 = E\Big(B(u^k), E\Big[H(u^k)\Delta_{k+1}W|\mathcal{F}_{t_k}\Big]\Big)_2 = 0. \qquad (6)$$

For $\alpha=\frac{1}{2}$, by using the Young inequality and the Itô isometry, we get

$$\begin{aligned}
E(H(u^k)\Delta_{k+1}W, B(u^{k+1}))_2 &= E(H(u^k)\Delta_{k+1}W, B(u^{k+1})-B(u^k))_2\\
&\leq E(\|H(u^k)\Delta_{k+1}W\|_2\cdot\|B(u^{k+1})-B(u^k)\|_2)\\
&\leq \frac{1}{2}\Big(\frac{1}{\alpha}E\|\int_{t_k}^{t_{k+1}} H(u^k)\,dW\|_2^2+\alpha E\|B(u^{k+1})-B(u^k)\|_2^2\Big)\\
&= E\int_{t_k}^{t_{k+1}}\|H(u^k)\|^2_{HS(L^2(D))}\,dt+\frac{1}{4}E\|B(u^{k+1})-B(u^k)\|_2^2\\
&= \tau E\|H(u^k)\|^2_{HS(L^2(D))}+\frac{1}{4}E\|B(u^{k+1})-B(u^k)\|_2^2. \qquad (7)
\end{aligned}$$

From (3) - (7), (H2) and (B1) we obtain the following inequality:

$$\begin{aligned}
&\frac{1}{2}E\|B(u^{k+1})\|_2^2-\frac{1}{2}E\|B(u^k)\|_2^2+\frac{1}{4}E\|B(u^{k+1})-B(u^k)\|_2^2+c_1c\tau\int_D|\nabla u^{k+1}|^p\,dx\\
\leq&\tau E\|H(u^k)\|^2_{HS(L^2(D))}+\tilde{c}\tau\|k_1\|_1\leq\tau C_1E\|u^k\|_2^2+\tilde{c}\tau\|k_1\|_1\\
\leq&\tau\frac{C_1}{c^2}E\|B(u^k)\|_2^2+\tilde{c}\tau\|k_1\|_1.
\end{aligned}$$

Now we sum over $k=0,...,N-1$ and get

$$\begin{aligned}
&\frac{1}{2}E\|B(u^n)\|_2^2-\frac{1}{2}\|B(u_0)\|_2^2+\frac{1}{4}E\sum_{k=0}^{n-1}\|B(u^{k+1})-B(u^k)\|_2^2+c_1c\tau E\sum_{k=0}^{n-1}\|\nabla u^{k+1}\|_p^p\\
\leq&\tilde{c}\tau\sum_{k=0}^{n-1}\|k_1\|_1+\frac{C_1}{c^2}\tau E\sum_{k=0}^{n-1}\|B(u^k)\|_2^2\leq\tilde{c}T\|k_1\|_1+\frac{C_1}{c^2}\tau E\sum_{k=0}^{n-1}\|B(u^k)\|_2^2.
\end{aligned}$$

□

Definition 4.2.2. *We define*

$$
\begin{aligned}
u_N(t) &:= \sum_{k=0}^{N-1} u^{k+1}\chi_{[t_k,t_{k+1})}(t),\ t\in[0,T),\ u_N(T)=u^N,\\
u_\tau(t) &:= \sum_{k=0}^{N-1} u^{k}\chi_{(t_k,t_{k+1}]}(t),\ t\in(0,T],\ u_\tau(0)=u_0,\\
B(u_N(t)) &:= \sum_{k=0}^{N-1} B(u^{k+1})\chi_{[t_k,t_{k+1})}(t),\ t\in[0,T),\ B(u_N(T))=B(u^N),\\
B(u_\tau(t)) &:= \sum_{k=0}^{N-1} B(u^{k})\chi_{(t_k,t_{k+1}]}(t),\ t\in(0,T],\ B(u_\tau(0))=B(u_0),\\
M_N(t) &:= \int_0^t H(u_\tau)\ dW,\ t\in[0,T],\\
\tilde{u}_N(t) &:= \sum_{k=0}^{N-1}\left(\frac{u^{k+1}-u^k}{\tau}(t-t_k)+u^k\right)\chi_{[t_k,t_{k+1})}(t),\ t\in[0,T),\ \tilde{u}_N(T)=u^N,\\
\tilde{B}_N(t) &:= \sum_{k=0}^{N-1}\left(\frac{B(u^{k+1})-B(u^k)}{\tau}(t-t_k)+B(u^k)\right)\chi_{[t_k,t_{k+1})}(t),\ t\in[0,T),\\
\tilde{B}_N(T) &= B(u^N),\\
\tilde{M}_N(t) &:= \sum_{k=0}^{N-1}\left(\frac{M_N(t_{k+1})-M_N(t_k)}{\tau}(t-t_k)+M_N(t_k)\right)\chi_{[t_k,t_{k+1})}(t),\ t\in[0,T),\\
\tilde{M}_N(T) &= M_N(T).
\end{aligned}
$$

Lemma 4.2.3. *There exists a constant $K>0$ such that for all $N\in\mathbb{N}$*

$$
\max_{n=1,\dots,N} E\|B(u^n)\|_2^2 = \max_{t\in[0,T]} E\|\tilde{B}_N(t)\|_2^2 \le K,
$$

$$
E\sum_{k=0}^{N-1}\|B(u^{k+1})-B(u^k)\|_2^2 \le K,\ E\int_0^T \|H(u_\tau)\|_{HS(L^2(D))}^2\ dt \le K,
$$

$$
E\int_0^T\int_D |\nabla u_N|^p\ dxdt \le K,\ E\int_0^T\int_D |A(\nabla u_N)|^{p'}\ dxdt \le K,
$$

$$
E\int_0^T\int_D |\nabla B(u_N)|^p\ dxdt \le K.
$$

Proof. We take the inequality in Lemma 4.2.1 and discard some nonnegative terms. Thereby we get

$$
\frac{1}{2}E\|B(u^n)\|_2^2 \le \frac{1}{2}\|B(u_0)\|_2^2 + \tilde{c}T\|k_1\|_1 + \tau\frac{C_1}{c^2}\sum_{k=0}^{n-1}E\|B(u^k)\|_2^2.
$$

Using the discrete Gronwall inequality we obtain for all $n = 1, ..., N$

$$E\|B(u^n)\|_2^2 \leq (\|B(u_0)\|_2^2 + 2\tilde{c}T\|k_1\|_1) \exp\left(\frac{2TC_1}{c^2}\right).$$

If we keep the term $\sum_{k=0}^{N-1} \|B(u^{k+1}) - B(u^k)\|_2^2$ in Lemma 4.2.1 we can see that this term is bounded. By using the same argument we may conclude that $E\int_0^T \int_D |\nabla u_N|^p \, dxdt$ is bounded, by (A3) it follows that also $E\int_0^T \int_D |A(\nabla u_N)|^{p'} \, dxdt$ is bounded and by (B1)

$$E\int_0^T \int_D |\nabla B(u_N)|^p \, dxdt = E\int_0^T \int_D |b'(u_N)|^p |\nabla u_N|^p \, dxdt \leq \tilde{c}^p E\int_0^T \int_D |\nabla u_N|^p \, dxdt$$

is bounded. Finally, the inequality

$$\|H(u_\tau(t))\|_{HS(L^2(D))}^2 \leq C_1\|u_\tau(t)\|_2^2 \leq \frac{C_1}{c^2}\|B(u_\tau(t))\|_2^2$$

for all $t \in [0, T]$ yields that $E\int_0^T \|H(u_\tau)\|_{HS(L^2(D))}^2 \, dt$ is bounded. □

Lemma 4.2.4. *There exists a constant $K > 0$ such that for all $N \in \mathbb{N}$*

$$E \max_{n=1,...,N} \|B(u^n)\|_2^2 = E \max_{t\in[0,T]} \|\tilde{B}_n(t)\|_2^2 \leq K.$$

Proof. Similar to the proof of Lemma 4.2.1 we get

$$\begin{aligned} &\frac{1}{2}(\|B(u^{k+1})\|_2^2 - \|B(u^k)\|_2^2 + \|B(u^{k+1}) - B(u^k)\|_2^2) + \tau\int_D A(\nabla u^{k+1}) \cdot \nabla B(u^{k+1}) \, dx \\ &= (H(u^k)\Delta_{k+1}W, B(u^{k+1}) - B(u^k))_2 + (H(u^k)\Delta_{k+1}W, B(u^k))_2 \qquad (8) \\ &\leq \frac{1}{2}\|H(u^k)\Delta_{k+1}W\|_2^2 + \frac{1}{2}\|B(u^{k+1}) - B(u^k)\|_2^2 + (H(u^k)\Delta_{k+1}W, B(u^k))_2. \end{aligned}$$

From (8) it follows that

$$\|B(u^{k+1})\|_2^2 - \|B(u^k)\|_2^2 - 2\tilde{c}\tau\|k_1\|_1 \leq \|H(u^k)\Delta_{k+1}W\|_2^2 + 2(H(u^k)\Delta_{k+1}W, B(u^k))_2$$

and hence after summing over $k = 0, ..., n-1$

$$\|B(u^n)\|_2^2 - \|B(u_0)\|_2^2 - 2\tilde{c}T\|k_1\|_1 \leq \sum_{k=0}^{n-1} \|H(u^k)\Delta_{k+1}W\|_2^2 + 2\sum_{k=0}^{n-1}(H(u^k)\Delta_{k+1}W, B(u^k))_2.$$

Now we take the maximum over $n = 1, ..., N$ and the expectation:

$$\begin{aligned} E \max_{n=1,...,N} \|B(u^n)\|_2^2 \leq \|B(u_0)\|_2^2 + 2\tilde{c}T\|k_1\|_1 + E\sum_{k=0}^{N-1} \|H(u^k)\Delta_{k+1}W\|_2^2 \\ + 2E \max_{n=1,...,N} \sum_{k=0}^{n-1}(H(u^k)\Delta_{k+1}W, B(u^k))_2. \end{aligned}$$

The rest of the proof is the same as in [14], p.90-92, if one replaces u_τ by $B(u_\tau)$. □

Lemma 4.2.5. *There exists a constant $K > 0$ such that for all $N \in \mathbb{N}$*

$$E\int_0^T \|\frac{d}{dt}(\tilde{B}_N - \tilde{M}_N)\|^{p'}_{W^{-1,p'}(D)}\ dt \leq K.$$

Proof. By considering the discrete equation we obtain

$$\frac{d}{dt}(\tilde{B}_N - \tilde{M}_N) = \text{div } A(\nabla u_N)$$

in $W^{-1,p'}(D)$, a.s. in Ω and

$$E\int_0^T \|\text{div } A(\nabla u_N)\|^{p'}_{W^{-1,p'}(D)}\ dt = E\int_0^T \sup_{\|\varphi\|_{W_0^{1,p}(D)}\leq 1} |\int_D |A(\nabla u_N)\cdot\nabla\varphi\ dx|^{p'} dt$$

$$\leq E\int_0^T \sup_{\|\varphi\|_{W_0^{1,p}(D)}\leq 1} \|\nabla\varphi\|_p^{p'}\cdot\|A(\nabla u_N)\|_{p'}^{p'}\ dt \leq E\int_0^T \|A(\nabla u_N)\|_{p'}^{p'}\ dt,$$

which is bounded since Lemma 4.2.3 holds true. □

Lemma 4.2.6. *Let $\mathcal{K}, \mathcal{H}$ be separable Hilbert spaces and Φ_k an $\mathcal{F}_{t_k}$-measurable random variable with values in $HS(\mathcal{K}, \mathcal{H})$. We define the left-continuous, $\mathcal{F}_t$-adapted process*

$$\Phi_\tau := \sum_{k=0}^{N-1} \Phi_k \chi_{(t_k,t_{k+1}]}.$$

Then, for any $p > 2$ there exist constants $\gamma > 0$ and $C_\gamma > 0$ and an integrable, real-valued random variable X only depending on γ such that

$$\sup_{k=0,\dots,N-1}\ \sup_{s\in[t_k,t_{k+1}]} \|\int_{t_k}^s \Phi_\tau\ dW\|_{\mathcal{H}}$$
$$\leq C_\gamma \tau^\gamma \left(\sup_{k=0,\dots,N-1} \tau\|\Phi_k\|^p_{HS(\mathcal{K},\mathcal{H})} + 1 + X \right).$$

Moreover, there exists a constant $C \geq 0$ such that

$$E(X) \leq C tr(Q).$$

Proof. We combine the Garsia-Rodemich-Rumsey inequality (see [7]) with the same arguments as in [14], p. 94-95. □

4.3 Regularity of approximate solutions

Lemma 4.3.1. *There exists a constant $K_1 > 0$ such that*

$$E\int_0^T \|H(u_\tau)\|^p_{HS(L^2(D),H_0^1(D))}\ dt \leq K_1.$$

Proof. See [14], p.96. □

Definition 4.3.2. *Let V be a Banach space, $1 < p < \infty$ and $0 < \alpha < 1$. The fractional Sobolev space $W^{\alpha,p}(0,T;V)$ is defined as follows (see [1]):*

$$W^{\alpha,p}(0,T;V) := \{f \in L^p(0,T;V); \|f\|_{W^{\alpha,p}(0,T;V)} < \infty\}$$

where

$$\|f\|_{W^{\alpha,p}(0,T;V)} := \left(\int_0^T \int_0^T \frac{\|f(t)-f(r)\|_V^p}{|t-r|^{\alpha p+1}}\ dtdr\right)^{\frac{1}{p}}.$$

Lemma 4.3.3. *For any $\alpha \in (0, \frac{1}{2})$ there exists a constant $C(\alpha,p) \geq 0$ such that*

$$E\|\int_0^\cdot H(u_\tau)\ dW\|^p_{W^{\alpha,p}(0,T;H_0^1(D))} \leq C(\alpha,p)K_1.$$

In particular, $\int_0^\cdot H(u_\tau)\ dW$ is bounded in $L^p(\Omega, W^{\alpha,p}(0,T;H_0^1(D)))$.

Proof. The assertion follows from [6], Lemma 2.1., p.369 and Lemma 4.3.1. □

Lemma 4.3.4. *$(\tilde{M}_N)$ is bounded in $L^p(\Omega, W^{\alpha,p}(0,T;H_0^1(D)))$ for any $\alpha \in (0,\gamma)$ and $\gamma = \frac{1}{2} - \frac{1}{p}$.*

Proof. The assertion follows from [2], Lemma 3.2, p.511 with the same arguments as in [14], p.97-99. □

Remark 4.3.5. *By the theorem of Lions-Aubin the space*

$$\mathcal{W} := \{v \in L^p(0,T;H_0^1(D)); \frac{d}{dt}v \in L^{p'}(0,T;W^{-1,p'}(D))\}$$

is compactly embedded into $\mathcal{C}([0,T];W^{-1,p'}(D))$ and compactly embedded into $L^2(0,T;L^2(D))$.

Lemma 4.3.6. *There exists a constant $C > 0$ such that*

$$\|\tilde{B}_N\|_{L^p(\Omega;L^p(0,T;W_0^{1,p}(D)))} + \|\tilde{B}_N - \tilde{M}_N\|_{L^{p'}(\Omega,\mathcal{W})} \leq C.$$

Proof. By an elementary calculation we may conclude that there exists a constant $\tilde{C} > 0$ such that

$$E\|\tilde{B}_N\|^p_{L^p(0,T;W_0^{1,p}(D))} \leq \tilde{C}E\tau\sum_{k=0}^N \|B(u^k)\|^p_{W_0^{1,p}(D)}$$

$$\leq \tilde{C}E\left(\int_0^T \int_D |\nabla B(u_N)|^p\ dxdt + T\|\nabla B(u_0)\|_p^p\right) \leq \tilde{C}(K + T\|\nabla B(u_0)\|_p^p).$$

From Lemma 4.3.4 it follows that $(\tilde{M}_N)$ is bounded in $L^p(\Omega, W^{\alpha,p}(0,T;H_0^1(D)))$, hence $(\tilde{B}_N - \tilde{M}_N)$ is bounded in $L^p(\Omega;L^p(0,T;H_0^1(D)))$. Now we apply Lemma 4.2.5 and the proof is complete. □

4.4 Tightness

We set $\mathcal{X} := C([0,T]; L^2(D)) \times L^2(0,T; L^2(D)) \times C([0,T]; U)$ and consider for all $N \in \mathbb{N}$ the image measures $\mu_{\tilde{B}_N} := P \circ (\tilde{B}_N)^{-1}$, $\mu_{M_N} := P \circ (M_N)^{-1}$ and $\mu_W := P \circ W^{-1}$. Their joint law in $\mathcal{X}$ is denoted by $\mu_N := (\mu_{\tilde{B}_N}, \mu_{W_N}, \mu_W)$. Then we have the following proposition:

Proposition 4.4.1. *The sequence $(\mu_{\tilde{B}_N})$ on $L^2(0,T; L^2(D))$ is tight, and the sequence (μ_{M_N}) on $\mathcal{C}([0,T]; L^2(D))$ is tight. As a constant sequence, the sequence (μ_W) on $\mathcal{C}([0,T]; U)$ is tight. In particular, the sequence (μ_N) on $\mathcal{X}$ is tight.*

Proof. The proof is the same as in [14], p. 100-101, if one replaces $\tilde{u}_N$ by $\tilde{B}_N$ and B_N by M_N. □

Remark 4.4.2. *Now we are able to use the theorem of Prokhorov (see [3], Theorem 5.1, p.59). It follows that the sequence (μ_N) is relatively compact, i.e., there exists a not relabeled subsequence of (μ_N) and a probability measure $\mu_\infty = (\mu_\infty^1, \mu_\infty^2, \mu_W)$ on $\mathcal{X}$ such that*

$$\lim_{N\to\infty} \int_{L^2(0,T;L^2(D))} \varphi \, d\mu_{\tilde{B}_N} = \lim_{N\to\infty} E[\varphi(\tilde{B}_N)] = \int_{L^2(0,T;L^2(D))} \varphi \, d\mu_\infty^1$$

for all $\varphi \in \mathcal{C}_b(L^2(0,T; L^2(D)))$ and

$$\lim_{N\to\infty} \int_{\mathcal{C}([0,T];L^2(D))} \psi \, d\mu_{M_N} = \lim_{N\to\infty} E[\psi(M_N)] = \int_{\mathcal{C}([0,T];L^2(D))} \psi \, d\mu_\infty^2$$

for all $\psi \in \mathcal{C}_b(\mathcal{C}([0,T]; L^2(D)))$.

4.5 Existence of martingale solutions

We apply the following version of the theorem of Skorokhod (see [12], Theorem 1.10.4, Addendum 1.10.5, p.59) to get the following proposition:

Proposition 4.5.1. *There exists a probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P})$ and a sequence of measurable functions $\phi_N : \hat{\Omega} \to \Omega$ such that $P = \hat{P} \circ (\phi_N)^{-1}$ for all $N \in \mathbb{N}$, and there exists a measurable function*

$$(B_\infty, M_\infty, W_\infty) : \hat{\Omega} \to \mathcal{X}$$

such that

i) $\hat{\tilde{B}}_N := \tilde{B}_N \circ \phi_N \to B_\infty$ *in* $L^2(0,T; L^2(D))$ *a.s. in* $\hat{\Omega}$,

ii) $\hat{M}_N := M_N \circ \phi_N \to M_\infty$ *in* $\mathcal{C}([0,T]; L^2(D))$ *a.s. in* $\hat{\Omega}$,

iii) $W_N := W \circ \phi_N \to W_\infty$ *in* $\mathcal{C}([0,T]; U)$ *a.s. in* $\hat{\Omega}$,

iv) $\mathcal{L}(B_\infty, M_\infty, W_\infty) = \mu_\infty$.

Definition 4.5.2. *For all $N \in \mathbb{N}$ we define*

$$
\begin{aligned}
v^k &:= u^k \circ \phi_N,\ k = 0, ..., N-1,\\
v_N(t) &:= \sum_{k=0}^{N-1} v^{k+1}\chi_{[t_k,t_{k+1})}(t),\ t \in [0,T),\ v_N(T) = v^N,\\
v_\tau(t) &:= \sum_{k=0}^{N-1} v^{k}\chi_{(t_k,t_{k+1}]}(t),\ t \in (0,T],\ v_\tau(0) = u_0,\\
B(v_N(t)) &:= \sum_{k=0}^{N-1} B(v^{k+1})\chi_{[t_k,t_{k+1})}(t),\ t \in [0,T),\ B(v_N(T)) = B(v^N),\\
\tilde{v}_N(t) &:= \sum_{k=0}^{N-1} \left(\frac{v^{k+1}-v^k}{\tau}(t-t_k) + v^k\right)\chi_{[t_k,t_{k+1})}(t),\ t \in [0,T),\ \tilde{v}_N(T) = v^N,\\
\hat{\tilde{M}}_N(t) &:= \sum_{k=0}^{N-1} \left(\frac{\hat{M}_N(t_{k+1})-\hat{M}_N(t_k)}{\tau}(t-t_k) + \hat{M}_N(t_k)\right)\chi_{[t_k,t_{k+1})}(t),\ t \in [0,T),\\
\hat{\tilde{M}}_N(T) &= \hat{M}_N(T).
\end{aligned}
$$

Lemma 4.5.3. *For all $N \in \mathbb{N}$, W_N is a Q-Wiener process with values in U adapted to the filtration $\mathcal{F}_t^{W_N} := \sigma(W_N(s))_{0\le s\le t}$.*

Proof. See [14], p. 103. □

Lemma 4.5.4. *For any $N \in \mathbb{N}$ and any $k = 0, ..., N-1$ we have*

$$B(v^{k+1}) - B(v^k) - \tau \text{ div } A(\nabla v^{k+1}) - H(v^k)\Delta_{k+1}W_N = 0 \tag{9}$$

a.s. in $\hat{\Omega}$.

Proof. For any $\hat{A} \in \hat{\mathcal{F}}$, by definition of the image measure and the fact that $P = \hat{P} \circ (\phi_N)^{-1}$ we obtain

$$
\begin{aligned}
&\int_{\hat{A}} B(v^{k+1}) - B(v^k) - \tau \text{ div } A(\nabla v^{k+1}) - H(v^k)\Delta_{k+1}W_N\ d\hat{P}\\
=&\int_{\phi_N(\hat{A})} B(u^{k+1}) - B(u^k) - \tau \text{ div } A(\nabla u^{k+1}) - H(u^k)\Delta_{k+1}W\ dP = 0.
\end{aligned}
$$

Hence $B(v^{k+1}) - B(v^k) - \tau \text{ div } A(\nabla v^{k+1}) - H(v^k)\Delta_{k+1}W_N = 0$ a.s. in $\hat{\Omega}$. □

Lemma 4.5.5. *We may conclude*

$$
\begin{aligned}
&\hat{M}_N(t) = \int_0^t H(v_\tau)\ dW_N,\ t \in [0,T],\\
&\hat{\tilde{B}}_N(t) = \sum_{k=0}^{N-1} \left(\frac{B(v^{k+1})-B(v^k)}{\tau}(t-t_k) + B(v^k)\right)\chi_{[t_k,t_{k+1})}(t),\ t \in [0,T),\ \hat{\tilde{B}}_N(T) = B(v^N).
\end{aligned}
$$

a.s. in $\hat{\Omega}$.

Proof. Since $v_\tau = u_\tau \circ \phi_N$, $W_N = W \circ \phi_N$ and $v^k = u^k \circ \phi_N$, the proof is a direct consequence of the definitions of $\hat{M}_N$ and $\hat{\tilde{B}}_N$. $\square$

Lemma 4.5.6. *There exists a constant $K > 0$ such that*

$$E \max_{n=1,\dots,N} \|B(v^n)\|_2^2 = E \max_{t\in[0,T]} \|\hat{\tilde{B}}_N(t)\|_2^2 \le K,$$
$$E \sum_{k=0}^{N-1} \|B(v^{k+1}) - B(v^k)\|_2^2 \le K, \; E \int_0^T \|H(v_\tau)\|^p_{HS(L^2(D),H_0^1(D))} \, dt \le K,$$
$$E \int_0^T \int_D |\nabla v_N|^p \, dxdt \le K, \; E \int_0^T \int_D |A(\nabla v_N)|^{p'} \, dxdt \le K,$$
$$E \int_0^T \int_D |\nabla B(v_N)|^p \, dxdt \le K.$$

Proof. We replace u^k by v^k for $k = 0, ..., N$ and repeat the arguments of Lemma 4.2.3, Lemma 4.2.4 and Lemma 4.3.1. $\square$

Lemma 4.5.7. *We have*

$i)$ $\hat{M}_N \to M_\infty$ *in* $L^q(\hat{\Omega}; \mathcal{C}([0,T]; L^2(D)))$ *for all* $1 \le q < p$,

$ii)$ $\hat{M}_N \rightharpoonup M_\infty$ *in* $L^p(\hat{\Omega}; W^{\alpha,p}(0,T; H_0^1(D)))$,

$iii)$ $\hat{\tilde{B}}_N \to B_\infty$ *in* $L^q(\hat{\Omega}; L^2(0,T; L^2(D)))$ *for all* $1 \le q < p$,

$iv)$ $B(v_N) \to B_\infty$ *in* $L^2(\hat{\Omega}; L^2(0,T; L^2(D)))$,

$v)$ $\hat{\tilde{B}}_N \rightharpoonup^* B_\infty$ *in* $L^2_w(\hat{\Omega}; L^\infty(0,T; L^2(D))) = L^2(\hat{\Omega}; L^1(0,T; L^2(D)))^*$.

Proof. See [14], p. 105-107, and replace $\hat{u}_N$ by $\hat{\tilde{B}}_N$, v_N by $B(v_N)$ and u_∞ by B_∞. $\square$

Remark 4.5.8. *By assumption (B1) it is easy to see that $B : L^2(\hat{\Omega} \times Q_T) \to L^2(\hat{\Omega} \times Q_T)$ is Lipschitz continuous and strongly monotone. Hence by the theorem of Zarantonello it is bijective with Lipschitz continuous inverse $B^{-1} : L^2(\hat{\Omega} \times Q_T) \to L^2(\hat{\Omega} \times Q_T)$.*
So we can set $v_\infty := B^{-1}(B_\infty) \in L^2(\hat{\Omega} \times Q_T)$.

Lemma 4.5.9. *Consider v_∞ of Remark 4.5.8. Then we have*

$i)$ $v_N \to v_\infty$ *in* $L^2(\hat{\Omega} \times Q_T)$,

$ii)$ $\tilde{v}_N \to v_\infty$ *in* $L^2(\hat{\Omega} \times Q_T)$,

$iii)$ $v_\tau \to v_\infty$ *in* $L^2(\hat{\Omega} \times Q_T)$,

$iv)$ $H(v_\tau) \to H(v_\infty)$ *in* $L^2(\hat{\Omega} \times (s,t); HS(L^2(D)))$ *for all* $0 \le s < t \le T$.

Proof. i): This is a direct consequence of Lemma 4.5.7 iv) and the continuity of B^{-1}.
ii): By Lemma 4.5.6 and assumption (B1) we can calculate

$$E\int_0^T \|\tilde{v}_N(t)-v_N(t)\|_2^2\,dt = E\sum_{k=0}^{N-1}\int_{t_k}^{t_{k+1}} \|\frac{v^{k+1}-v^k}{\tau}(t-t_k)+v^k-v^{k+1}\|_2^2\,dt$$

$$=E\sum_{k=0}^{N-1}\|v^{k+1}-v^k\|_2^2\int_{t_k}^{t_{k+1}}(\frac{t-t_k}{\tau}-1)^2\,dt = \frac{\tau}{3}E\sum_{k=0}^{N-1}\|v^{k+1}-v^k\|_2^2$$

$$\leq\frac{\tau}{3c^2}E\sum_{k=0}^{N-1}\|B(v^{k+1})-B(v^k)\|_2^2 \leq \frac{\tau}{3c^2}K\to 0.$$

iii): Again we use Lemma 4.5.6 and assumption (B1) to get

$$E\int_0^T \|v_\tau(t)-v_N(t)\|_2^2\,dt = E\sum_{k=0}^{N-1}\int_{t_k}^{t_{k+1}}\|v^{k+1}-v^k\|_2^2$$

$$\leq\frac{\tau}{c^2}E\sum_{k=0}^{N-1}\int_{t_k}^{t_{k+1}}\|B(v^{k+1})-B(v^k)\|_2^2 \leq \frac{\tau}{c^2}K\to 0.$$

iv) We use assumption (H1) und Lemma 4.5.9 ii) and obtain

$$E\int_s^t \|H(v_\tau)-H(v_\infty)\|_{HS(L^2(D))}^2\,dt = E\int_s^t\sum_{n=1}^\infty \|h_n(v_\tau)-h_n(v_\infty)\|_2^2\,dt$$

$$\leq C_1 E\int_s^t \|v_\tau-v_\infty\|_2^2\to 0.$$

□

Remark 4.5.10. *By definition (see, e.g., [5])* $B_\infty\in L^\infty(0,T;L^2(D))$ *a.s. in* $\hat{\Omega}$.

Lemma 4.5.11. *For a not relabeled subsequence we have*

$$\nabla v_N \rightharpoonup \nabla v_\infty \text{ in } L^p(\hat{\Omega}\times Q_T)^d,$$
$$\nabla B(v_N) \rightharpoonup \nabla B_\infty \text{ in } L^p(\hat{\Omega}\times Q_T)^d.$$

There exists a function $G\in L^{p'}(\hat{\Omega}\times Q_T)^d$ *such that*

$$A(\nabla v_N)\rightharpoonup G \text{ in } L^{p'}(\hat{\Omega}\times Q_T)^d.$$

Proof. v_N and $B(v_N)$ are bounded in $L^p(\hat{\Omega};L^p(0,T;W_0^{1,p}(D)))$, hence there exist functions $f,g\in L^p(\hat{\Omega}\times Q_T)^d$ such that

$$v_N\rightharpoonup f \text{ in } L^p(\hat{\Omega};L^p(0,T;W_0^{1,p}(D))),$$
$$B(v_N)\rightharpoonup g \text{ in } L^p(\hat{\Omega};L^p(0,T;W_0^{1,p}(D))).$$

We have $L^p(\hat\Omega; L^p(0,T; W_0^{1,p}(D))) \hookrightarrow L^2(\hat\Omega \times Q_T)$, hence by Lemma 4.5.7 and Lemma 4.5.9 we may conclude $f = v_\infty$ and $g = B_\infty$. Thus

$$\nabla v_N \rightharpoonup \nabla v_\infty \text{ in } L^p(\hat\Omega \times Q_T)^d,$$
$$\nabla B(v_N) \rightharpoonup \nabla B_\infty \text{ in } L^p(\hat\Omega \times Q_T)^d.$$

In particular, $A(\nabla v_N)$ is bounded in $L^{p'}(\hat\Omega \times Q_T)^d$, hence the existence of a function G as claimed in the lemma is clear. □

Lemma 4.5.12. *There exist constants $\gamma > 0$, $C > 0$ and $C_\gamma > 0$ such that*

$$E \sup_{t\in[0,T]} \|\hat M_N(t) - \hat{\hat M}_N(t)\|_{H_0^1(D)} \le 2C_\gamma \tau^\gamma \left(E \int_0^T \|H(v_\tau)\|^p_{HS(L^2(D),H_0^1(D))} \, dt + 1 + C tr(Q) \right).$$

In particular, by Lemma 4.5.6

$$\lim_{N\to\infty} E \sup_{t\in[0,T]} \|\hat M_N(t) - \hat{\hat M}_N(t)\|_{H_0^1(D)} = 0.$$

Proof. See [14], p. 107-108. □

Proposition 4.5.13. *$v_\infty : \hat\Omega \times [0,T] \to L^2(D)$ is a stochastic process with $v_\infty(0) = u_0$ such that*

$$B(v_\infty(t)) = B(u_0) + \int_0^t \operatorname{div} G \, ds + M_\infty(t) \tag{10}$$

in $L^2(D)$, a.s. in $\hat\Omega$, for all $t \in [0,T]$.

Proof. This proof can be done analogously to the proof in [14], p. 108-112. A similar argumentation leads to

$$\frac{d}{dt}(B_\infty - M_\infty) = \operatorname{div} G$$

in $L^{p'}(\hat\Omega; L^{p'}(0,T; W^{-1,p'}(D)))$. We see that $B_\infty, M_\infty \in \mathcal{C}([0,T]; L^2(D))$ a.s. in $\hat\Omega$, hence $B_\infty(t) \in L^2(D)$ makes sense for all $t \in [0,T]$, a.s. in $\hat\Omega$. Then we can show that $\hat{\hat B}_N(t) \rightharpoonup B_\infty(t)$ in $L^2(\hat\Omega \times D)$ for all $t \in [0,T]$, and in particular we get $B_\infty(0) = B(u_0)$. □

Corollary 4.5.14.

$$\hat{\hat B}_N(t) \rightharpoonup B_\infty(t)$$

in $L^2(\hat\Omega \times D)$ for all $t \in [0,T]$.

Now, the following lemma ends the proof of Proposition 4.5.13:

Lemma 4.5.15. *v_∞ is a stochastic process with values in $L^2(D)$.*

Proof. Since $B^{-1} : L^2(D) \to L^2(D)$ is continuous, it is sufficient to prove that B_∞ is a stochastic process with values in $L^2(D)$. The proof of this result is similar to the proof in [14], p. 112, if one replaces u_∞ by B_∞. □

Proposition 4.5.16. *M_∞ is an $\mathcal{F}_t^\infty$-martingale, where $(\mathcal{F}_t^\infty)$ is the augmentation of the filtration $\hat{\mathcal{F}}_t^\infty := \sigma(M_\infty(s), v_\infty(s), W_\infty(s))_{0\le s\le t}$, $t \in [0,T]$, i.e., $\mathcal{F}_t^\infty$ is the smallest complete and right-continuous filtration containing $(\hat{\mathcal{F}}_t^\infty)$. The quadratic variation process of M_∞ is*

$$\ll M_\infty \gg_t = \int_0^t (H(v_\infty) \circ Q^{\frac{1}{2}}) \circ (H(v_\infty) \circ Q^{\frac{1}{2}})^* \, ds$$

for all $t \in [0,T]$.

Proof. See [14], p. 113-117. Since we want to avoid the use of the Martingale Representation Theorem in the sequel, we add W_∞ to the limit filtration (see [14], p.127-128). □

Lemma 4.5.17. *W_∞ is an $\mathcal{F}_t^\infty$-martingale.*

Proof. See [14], p. 128-129. □

Lemma 4.5.18. *W_∞ is a Q-Wiener process in U, adapted to $\mathcal{F}_t^\infty$ with increments $W_\infty(t) - W_\infty(s)$, $0 \le s \le t \le T$, independent of $\mathcal{F}_s^\infty$.*

Proof. With similar arguments as in [14], p. 129, we use the generalized version of Levy's Theorem (see [4], Theorem 4.4, p.89). □

Corollary 4.5.19. *We define*

$$M(t) = \int_0^t H(v_\infty) \, dW_\infty.$$

Then M is an $\mathcal{F}_t^\infty$-martingale with quadratic variation process

$$\ll M \gg_t = \int_0^t (H(v_\infty) \circ Q^{\frac{1}{2}}) \circ (H(v_\infty) \circ Q^{\frac{1}{2}})^* \, ds$$

for all $t \in [0,T]$.

Lemma 4.5.20. *We have the cross quadratic variation*

$$\ll W_\infty, M_\infty \gg_t = \int_0^t (H(v_\infty) \circ Q)^* \, ds.$$

Proof. We apply the results of [9], Theorem 3.12, p.12 in the same way as in [14], p. 130. □

Lemma 4.5.21. *For all $t \in [0,T]$ we have*

$$\ll M - M_\infty \gg_t = 0.$$

From this equality we get $M(t) = M_\infty(t)$ for all $t \in [0,T]$, a.s. in $\hat{\Omega}$.

Proof. See [14], p. 131. □

Lemma 4.5.22. *We have $G = A(\nabla v_\infty)$ in $L^{p'}(\hat{\Omega} \times Q_T)^d$.*

Proof. Testing the discrete equation (9) in Lemma 4.5.4 with $B(v^{k+1})$ we get

$$\begin{aligned}&(B(v^{k+1}) - B(v^k), B(v^{k+1}))_2 + \tau \int_D A(\nabla v^{k+1}) \cdot \nabla B(v^{k+1})\, dx\\ =&(\hat{M}_N(t_{k+1}) - \hat{M}_N(t_k), B(v^{k+1}))_2.\end{aligned}$$

It follows that

$$\begin{aligned}&\frac{1}{2}\left(\|B(v^{k+1})\|_2^2 - \|B(v^k)\|_2^2 + \|B(v^{k+1}) - B(v^k)\|_2^2\right) + \tau \int_D A(\nabla v^{k+1}) \cdot \nabla B(v^{k+1})\, dx\\ =&(\hat{M}_N(t_{k+1}) - \hat{M}_N(t_k), B(v^{k+1}))_2.\end{aligned}$$

Using the same argument as in Lemma 4.2.1 we can see that

$$E(\hat{M}_N(t_{k+1}) - \hat{M}_N(t_k), B(v^k))_2 = 0,$$

so after taking the expectation and summing over $k = 0, ..., N-1$ we obtain:

$$\begin{aligned}&\frac{1}{2}E\|B(v^N)\|_2^2 - \frac{1}{2}\|B(u_0)\|_2^2 + \frac{1}{2}\sum_{k=0}^{N-1} E\|B(v^{k+1}) - B(v^k)\|_2^2\\ &+\tau \sum_{k=0}^{N-1} E\int_D A(\nabla v^{k+1}) \cdot \nabla B(v^{k+1})\, dx\\ =&\sum_{k=0}^{N-1} E(\hat{M}_N(t_{k+1}) - \hat{M}_N(t_k), B(v^{k+1}) - B(v^k))_2\\ \le&\frac{1}{2}\sum_{k=0}^{N-1}\left(E\|\hat{M}_N(t_{k+1}) - \hat{M}_N(t_k)\|_2^2 + E\|B(v^{k+1}) - B(v^k)\|_2^2\right).\end{aligned}$$

It follows that

$$\begin{aligned}&\frac{1}{2}E\|B(v^N)\|_2^2 - \frac{1}{2}\|B(u_0)\|_2^2 + E\int_0^T \int_D b'(v_N)A(\nabla v_N) \cdot \nabla v_N\, dxdt\\ \le&\frac{1}{2}\sum_{k=0}^{N-1} \|\int_{t_k}^{t_{k+1}} H(v_\tau)\, dW_N\|_2^2 = \frac{1}{2}\sum_{k=0}^{N-1}\int_{t_k}^{t_{k+1}} \|H(v_\tau)\|_{HS(L^2(D))}^2\, dt\\ =&\frac{1}{2}\int_0^T \|H(v_\tau)\|_{HS(L^2(D))}^2\, dt.\end{aligned}$$

Hence we may conclude the inequality

$$\begin{aligned}\frac{1}{2}\|B(u_0)\|_2^2 \ge& \frac{1}{2}E\|B(v^N)\|_2^2 + E\int_0^T \int_D b'(v_N)A(\nabla v_N) \cdot \nabla v_N\, dxdt\\ &- \frac{1}{2}\int_0^T \|H(v_\tau)\|_{HS(L^2(D))}^2\, dt. \qquad (11)\end{aligned}$$

With the information that $M_\infty(t) = \int_0^t H(v_\infty)\, dW$ we are now in the position to use the Itô formula for the limit equation (10) in Proposition 4.5.13 with the functional $\frac{1}{2}\|\cdot\|_2^2 : L^2(D) \to \mathbb{R}$ (see [10], p.75, Theorem 4.2.5). Taking the expectation we get

$$\frac{1}{2}\|B(u_0)\|_2^2 = \frac{1}{2}E\|B(v_\infty(T))\|_2^2 + E\int_0^T \int_D G\cdot\nabla B(v_\infty)\, dxdt - \frac{1}{2}E\int_0^T \|H(v_\infty)\|^2_{HS(L^2(D))}\, dt. \tag{12}$$

From equation (12) and the inequality (11) we obtain

$$\begin{aligned}&\frac{1}{2}E\|B(v^N)\|_2^2 + E\int_0^T\int_D b'(v_N)A(\nabla v_N)\cdot\nabla v_N\, dxdt - \frac{1}{2}E\int_0^T \|H(v_\tau)\|^2_{HS(L^2(D))}\, dt\\ \leq&\frac{1}{2}E\|B(v_\infty(T))\|_2^2 + E\int_0^T\int_D G\cdot\nabla B(v_\infty)\, dxdt - \frac{1}{2}E\int_0^T \|H(v_\infty)\|^2_{HS(L^2(D))}\, dt.\end{aligned}$$

It follows that

$$\begin{aligned}&E\int_0^T\int_D b'(v_N)A(\nabla v_N)\cdot\nabla v_N\, dxdt - \frac{1}{2}E\int_0^T \|H(v_\tau)\|^2_{HS(L^2(D))}\, dt\\ &-E\int_0^T\int_D G\cdot\nabla B(v_\infty)\, dxdt + \frac{1}{2}E\int_0^T \|H(v_\infty)\|^2_{HS(L^2(D))}\, dt\\ \leq&\frac{1}{2}E\|B(v_\infty(T))\|_2^2 - \frac{1}{2}E\|B(v^N)\|_2^2\end{aligned} \tag{13}$$

We know that the equation $B(v^N) = \hat{\tilde{B}}_N(T)$ holds true, and from Corollary 4.5.14 we obtain $\hat{\tilde{B}}_N(T) \rightharpoonup B(v_\infty(T))$ in $L^2(\hat{\Omega}\times D)$. Since $\|\cdot\|_2^2$ is weakly lower semi-continuous we have

$$E\|B(v_\infty(T))\|_2^2 \leq \liminf_{N\to\infty} E\|\hat{\tilde{B}}_N(T)\|_2^2$$

and hence

$$\limsup_{N\to\infty}\left(E\|B(v_\infty(T))\|_2^2 - E\|\hat{\tilde{B}}_N(T)\|_2^2\right) \leq 0. \tag{14}$$

By Lemma 4.5.9 iv) we get

$$H(v_\tau) \to H(v_\infty)$$

in $L^2(\hat{\Omega}\times(0,T); HS(L^2(D)))$, hence $E\int_0^T \|H(v_\tau)\|^2_{HS(L^2(D)}\, dt \to E\int_0^T \|H(v_\infty)\|^2_{HS(L^2(D)}\, dt$. With this information, from (13) and (14) we may conclude the following inequality:

$$\limsup_{N\to\infty}\left(E\int_0^T\int_D b'(v_N)A(\nabla v_N)\cdot\nabla v_N\, dxdt - E\int_0^T\int_D G\cdot\nabla B(v_\infty)\, dxdt\right) \leq 0. \tag{15}$$

Let us consider

$$E\int_0^T\int_D b'(v_N)A(\nabla v_N)\cdot\nabla v_N\ dxdt - E\int_0^T\int_D G\cdot\nabla B(v_\infty)\ dxdt = I_1+I_2,$$

where

$$I_1 := E\int_0^T\int_D b'(v_N)(A(\nabla v_N)-G)\cdot\nabla v_N\ dxdt,$$
$$I_2 := E\int_0^T\int_D G\cdot(\nabla B(v_N)-\nabla B(v_\infty))\ dxdt.$$

Since $\nabla B(v_N)\rightharpoonup\nabla B(v_\infty)$ in $L^p(\hat\Omega\times Q_T)^d$, it follows $\lim\limits_{N\to\infty} I_2=0$.
Now we write

$$I_1 = I_3+I_4,$$

where

$$I_3 := E\int_0^T\int_D b'(v_N)(A(\nabla v_N)\cdot\nabla v_N - G\cdot\nabla v_\infty)\ dxdt,$$
$$I_4 := E\int_0^T\int_D b'(v_N)G\cdot(\nabla v_\infty-\nabla v_N)\ dxdt.$$

For a not relabeled subsequence v_N converges to v_∞ a.e. in $\hat\Omega\times Q_T$. Since b' is continuous, $b'(v_n)\to b'(v_\infty)$ a.e. in $\hat\Omega\times Q_T$ and then $b'(v_n)f\to b'(v_\infty)f$ a.e. in $\hat\Omega\times Q_T$ for all $f\in L^q(\hat\Omega\times Q_T)^d$, $1<q<\infty$. Now we have $|b'(v_N)f|^q\le\tilde c^q|f|^q\in L^1(\hat\Omega\times Q_T)$, hence by the theorem of Lebesgue we obtain

$$b'(v_n)f\to b'(v_\infty)f \text{ in } L^q(\hat\Omega\times Q_T)^d.$$

Since $\nabla v_N\rightharpoonup\nabla v_\infty$ in $L^p(\hat\Omega\times Q_T)^d$, it follows $\lim\limits_{N\to\infty} I_4=0$ by taking $q=p'$ and $f=G$.
We write

$$I_3 = I_5+I_6+I_7,$$

where

$$I_5 := E\int_0^T\int_D b'(v_N)(A(\nabla v_N)-A(\nabla v_\infty))\cdot(\nabla v_N-\nabla v_\infty)\ dxdt,$$
$$I_6 := E\int_0^T\int_D b'(v_N)A(\nabla v_\infty)\cdot(\nabla v_N-\nabla v_\infty)\ dxdt,$$
$$I_7 := E\int_0^T\int_D b'(v_N)\nabla v_\infty\cdot(A(\nabla v_N)-G)\ dxdt.$$

If we take $q = p'$ and $f = A(\nabla v_\infty)$ we can see that $\lim_{N\to\infty} I_6 = 0$ and if we take $q = p$ and $f = \nabla v_\infty$ we have $\lim_{N\to\infty} I_7 = 0$.
Thanks to (B1) and since a is monotone we can calculate

$$I_5 \geq c \cdot E \int_0^T \int_D (A(\nabla v_N) - A(\nabla v_\infty)) \cdot (\nabla v_N - \nabla v_\infty)\ dxdt \geq 0.$$

Thus we obtain

$$\begin{aligned}
0 &\leq \limsup_{N\to\infty} \left(c \cdot E \int_0^T \int_D (A(\nabla v_N) - A(\nabla v_\infty)) \cdot (\nabla v_N - \nabla v_\infty)\ dxdt \right) \\
&= \limsup_{N\to\infty}\ I_5 = \limsup_{N\to\infty}\ (I_5 + I_6 + I_7) = \limsup_{N\to\infty}\ I_3 \\
&= \limsup_{N\to\infty}\ (I_3 + I_4) = \limsup_{N\to\infty}\ I_1 = \limsup_{N\to\infty}\ (I_1 + I_2) \\
&= \limsup_{N\to\infty} \left(E \int_0^T \int_D b'(v_N) A(\nabla v_N) \cdot \nabla v_N\ dxdt - E \int_0^T \int_D G \cdot \nabla B(v_\infty)\ dxdt \right) \leq 0.
\end{aligned}$$

Hence

$$\lim_{N\to\infty} \left(E \int_0^T \int_D (A(\nabla v_N) - A(\nabla v_\infty)) \cdot (\nabla v_N - \nabla v_\infty)\ dxdt \right) = 0.$$

Thus we get

$$\lim_{N\to\infty} E \int_0^T \int_D A(\nabla v_N) \cdot \nabla v_N\ dxdt = E \int_0^T \int_D G \cdot \nabla v_\infty\ dxdt.$$

Since $A : L^p(\hat{\Omega} \times Q_T)^d \to L^{p'}(\hat{\Omega} \times Q_T)^d$ fulfills the (M)-property we may conclude $A(\nabla v_\infty) = G$ in $L^{p'}(\hat{\Omega} \times Q_T)^d$. □

Now the proof of Theorem 3.4 is complete.

5 Uniqueness with respect to the same probability space

In this section we show the uniqueness of a solution provided these solutions are solutions with respect to the same probability space, the same filtration, the same initial value and the same cylindrical Wiener process.

Proposition 5.1. *Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ be a filtered probability space, $u_0 \in W_0^{1,p}(D)$ and W a cylindrical Wiener process as in Section 2.3. If u_1 and u_2 are two solutions with respect to $(\Omega, \mathcal{F}, (\mathcal{F}_t), P, u_0, W)$, then $u_1(t) = u_2(t)$ for all $t \in [0, T]$, a.e. in $\Omega \times D$.*

Proof. Let u_1 and u_2 be two solutions with respect to $(\Omega, \mathcal{F}, (\mathcal{F}_t), P, u_0, W)$. We show $B(u_1(t)) = B(u_2(t))$ for all $t \in [0, T]$, a.e. in $\Omega \times D$. Then the assertion follows since b

is bijective as a function from $\mathbb{R}$ to $\mathbb{R}$.
We consider a smooth approximation of the absolute value, more precisely: For $\delta > 0$ let

$$\eta_\delta(r) = \begin{cases} -r - \frac{3}{4}\delta, & \text{if } r < -2\delta, \\ -\frac{1}{64\delta^3}r^4 + \frac{3}{8\delta}r^2, & \text{if } |r| \leq 2\delta, \\ r - \frac{3}{4}\delta, & \text{if } r > 2\delta. \end{cases}$$

Then $\eta_\delta \in C^2(\mathbb{R})$, η_δ is convex, $\eta_\delta'(r) = 1$ for $r > 2\delta$ and $\eta_\delta'(r) = -1$ for $r < 2\delta$. It follows that η_δ is Lipschitz continuous with Lipschitz constant 1, η_δ'' has compact support $[-2\delta, 2\delta]$ and $0 \leq \eta_\delta'' \leq \frac{3}{4\delta}$.
Since u_1 and u_2 are both solutions of (P) we get the equation

$$B(u_1(t)) - B(u_2(t)) = \int_0^t \operatorname{div}\Big(A(\nabla u_1) - A(\nabla u_2)\Big)\, ds + \int_0^t H(u_1) - H(u_2)\, dW$$

in $L^2(D)$ for all $t \in [0, T]$, a.s. in Ω. Now we use a version of the Itô formula one can find in [9], p.65 for the function $\phi : L^2(D) \to \mathbb{R}$, $\phi(u) = \int_D \eta_\delta(u)\, dx$ (see [9], p.72-74, Example 4.1, Remark 4.2). If we do so, we get

$$I_1 + I_2 = I_3 + I_4,$$

where

$$\begin{aligned} I_1 &= \int_D \eta_\delta(B(u_1)(t) - B(u_2(t)))\, dx, \\ I_2 &= \int_0^t \int_D (A(\nabla u_1) - A(\nabla u_2)) \cdot (\nabla B(u_1) - \nabla B(u_2))\eta_\delta''(B(u_1) - B(u_2))\, dxds, \\ I_3 &= \int_0^t \Big(\eta_\delta(B(u_1) - B(u_2)), H(u_1) - H(u_2)\, dW\Big)_2, \\ I_4 &= \frac{1}{2}\int_0^t \int_D \eta_\delta''(B(u_1) - B(u_2)) \sum_{n=1}^{\infty} |h_n(u_1) - h_n(u_2)|^2\, dxds. \end{aligned}$$

We have $\eta_\delta(B(u_1)(t) - B(u_2(t))) \to |B(u_1)(t) - B(u_2(t))|$ for $\delta \to 0^+$ a.e. in $\Omega \times D$. Since $\eta_\delta(B(u_1)(t) - B(u_2(t))) \leq |B(u_1)(t) - B(u_2(t))|$ for all $\delta > 0$, by the theorem of Lebesgue we obtain:

$$\lim_{\delta \to 0^+} E(I_1) = E\int_D |B(u_1(t)) - B(u_2(t))|\, dx$$

for all $t \in [0, T]$.

Now we split the term I_2 in two terms as follows:

$$
\begin{aligned}
I_2 &= \int_0^t \int_D (A(\nabla u_1) - A(\nabla u_2)) \cdot (\nabla B(u_1) - \nabla B(u_2)) \eta_\delta''(B(u_1) - B(u_2))\; dxds \\
&= \int_0^t \int_D (A(\nabla u_1) - A(\nabla u_2)) \cdot (b'(u_1)\nabla u_1 - b'(u_2)\nabla u_2) \eta_\delta''(B(u_1) - B(u_2))\; dxds \\
&= \int_0^t \int_D b'(u_1)(A(\nabla u_1) - A(\nabla u_2)) \cdot (\nabla u_1 - \nabla u_2) \eta_\delta''(B(u_1) - B(u_2))\; dxds \\
&+ \int_0^t \int_D (b'(u_1) - b'(u_2))\nabla u_2 \cdot (A(\nabla u_1) - A(\nabla u_2)) \eta_\delta''(B(u_1) - B(u_2))\; dxds \\
&= I_2^1 + I_2^2.
\end{aligned}
$$

Since a is monotone, we can see that $I_2^1 \geq 0$. Now we can calculate

$$
|I_2^2| \leq \int_0^t \int_{\{|B(u_1)-B(u_2)| \leq 2\delta\}} \frac{3}{4\delta} |b'(u_1) - b'(u_2)||\nabla u_2||A(\nabla u_1) - A(\nabla u_2)|\; dxds.
$$

Since we have assumption (B1) we know that b' is Lipschitz continuous, hence there exists $L > 0$ such that

$$
|b'(u_1) - b'(u_2)| \leq L|u_1 - u_2| \leq \frac{L}{c}|B(u_1) - B(u_2)|.
$$

Now it follows

$$
\begin{aligned}
|I_2^2| &\leq \int_0^t \int_{\{|B(u_1)-B(u_2)| \leq 2\delta\}} \frac{3L}{4c\delta} |B(u_1) - B(u_2)||\nabla u_2||A(\nabla u_1) - A(\nabla u_2)|\; dxds \\
&\leq \int_0^t \int_{\{|B(u_1)-B(u_2)| \leq 2\delta\}} \frac{3L}{4c\delta} \cdot 2\delta |\nabla u_2||A(\nabla u_1) - A(\nabla u_2)|\; dxds \\
&= \int_0^t \int_{\{|B(u_1)-B(u_2)| \leq 2\delta\}} \frac{3L}{2c} |\nabla u_2||A(\nabla u_1) - A(\nabla u_2)|\; dxds.
\end{aligned}
$$

By the theorem of Lebesgue we get

$$
\begin{aligned}
E|I_2^2| &\leq E\int_0^t \int_{\{|B(u_1)-B(u_2)| \leq 2\delta\}} \frac{3L}{2c} |\nabla u_2||A(\nabla u_1) - A(\nabla u_2)|\; dxds \\
&\to E\int_{\{|B(u_1)-B(u_2)| = 0\}} \frac{3L}{2c} |\nabla u_2||A(\nabla u_1) - A(\nabla u_2)| \\
&= E\int_{\{u_1 = u_2\}} \frac{3L}{2c} |\nabla u_2||A(\nabla u_1) - A(\nabla u_2)|\; dxds = 0.
\end{aligned}
$$

Hence

$$
\liminf_{\delta \to 0^+} E(I_2) \geq 0.
$$

Since I_3 is a stochastic integral, we have $E(I_3) = 0$. It remains to consider I_4:

$$\begin{aligned}
|E(I_4)| &\le \frac{3}{4\delta} E \int_0^t \int_{\{|B(u_1)-B(u_2)|\le 2\delta\}} \sum_{n=1}^{\infty} |h_n(u_1) - h_n(u_2)|^2 \; dxds \\
&\le \frac{3C_1}{4\delta} E \int_0^t \int_{\{|B(u_1)-B(u_2)|\le 2\delta\}} |u_1 - u_2|^2 \; dxds \\
&\le \frac{3C_1}{4c^2\delta} E \int_0^t \int_{\{|B(u_1)-B(u_2)|\le 2\delta\}} |B(u_1) - B(u_2)|^2 \; dxds \\
&\le \frac{3C_1\delta}{c^2} E \int_0^t \int_{\{|B(u_1)-B(u_2)|\le 2\delta\}} 1 \; dxds \\
&\to 0
\end{aligned}$$

for $\delta \to 0^+$. Combining the previous estimates we have

$$E \int_D |B(u_1(t)) - B(u_2(t))| \; dx = \lim_{\delta\to 0^+} E(I_1) \le \liminf_{\delta\to 0^+} E(I_1 + I_2) = \liminf_{\delta\to 0^+} E(I_3 + I_4) = 0$$

for all $t \in [0, T]$. Thus we get $B(u_1(t)) = B(u_2(t))$ for all $t \in [0, T]$, a.e. in $\Omega \times D$. □

6 Existence and uniqueness of strong solutions

Now use the results of Section 3, 4 and 5 to show the existence and uniqueness of strong solutions. In fact, we will prove Theorem 3.5. To do so, we use the following lemma (see [8], Lemma 1.1):

Lemma 6.1. *Let V be a Polish space equipped with the Borel σ-algebra. A sequence of V-valued random variables (X_n) converges in probability if and only if for every pair of subsequences X_l and X_k there exists a joint subsequence (X_{l_j}, X_{k_j}) which converges for $j \to \infty$ in law to a probability measure μ such that*

$$\mu(\{(w, z) \in V \times V; \; w = z\}) = 1.$$

Let $(\tilde{B}_K, M_K, W)$ and $(\tilde{B}_L, M_L, W)$ be subsequences of $(\tilde{B}_N, M_N, W)$. Since

$$(\tilde{B}_K, M_K, W, \tilde{B}_L, M_L, W)$$

on

$$\mathcal{X} := \Big(L^2(0, T; L^2(D)) \times \mathcal{C}([0, T]; L^2(D)) \times \mathcal{C}([0, T]; U) \Big)^2,$$

is tight, this sequence is relatively compact by the theorem of Prokhorov, i.e., there exists a subsequence

$$(\tilde{B}_{K_j}, M_{K_j}, W_j, \tilde{B}_{L_j}, M_{L_j}, W_j)$$

which converges in law to a probability measure μ. By the theorem of Skorokhod there exists a probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P})$ and a sequence of measurable functions $\phi_j : \hat{\Omega} \to \Omega$, such that $P = \hat{P} \circ \phi_j^{-1}$ for all $j \in \mathbb{N}$ and measurable functions

$$(B^1_\infty, M^1_\infty, W_\infty, B^2_\infty, M^2_\infty, W_\infty) : \hat{\Omega} \to \mathcal{X}$$

satisfying the following properties:

$$\begin{aligned}
i)\ &\hat{\tilde{B}}_{K_j} := \tilde{B}_{K_j} \circ \phi_j \to B^1_\infty \text{ in } L^2(0,T;L^2(D)), \text{ a.s. in } \hat{\Omega},\\
ii)\ &\hat{\tilde{B}}_{L_j} := \tilde{B}_{L_j} \circ \phi_j \to B^2_\infty \text{ in } L^2(0,T;L^2(D)), \text{ a.s. in } \hat{\Omega},\\
iii)\ &\hat{M}_{K_j} := M_{K_j} \circ \phi_j \to M^1_\infty \text{ in } \mathcal{C}([0,T];L^2(D)), \text{ a.s. in } \hat{\Omega},\\
iv)\ &\hat{M}_{L_j} := M_{L_j} \circ \phi_j \to M^2_\infty \text{ in } \mathcal{C}([0,T];L^2(D)), \text{ a.s. in } \hat{\Omega},\\
v)\ &W_j := W \circ \phi_j \to W_\infty \text{ in } \mathcal{C}([0,T];U), \text{ a.s. in } \hat{\Omega},\\
vi)\ &\mathcal{L}(B^1_\infty, M^1_\infty, W_\infty, B^2_\infty, M^2_\infty, W_\infty) = \mu.
\end{aligned}$$

If we define $v^i_\infty := B^{-1}(B^i_\infty)$ for $i = 1, 2$ then by using the same argumentation as in Section 4 we can prove that the equation $M^i_\infty = \int_0^\cdot H(v^i_\infty)\, dW_\infty$ holds true and that for $i = 1, 2$, v^i_∞ are solutions of (P) with respect to $(\hat{\Omega}, \hat{\mathcal{F}}, (\mathcal{F}^\infty_t), \hat{P}, u_0, W_\infty)$, where $(\mathcal{F}^\infty_t)$ is the augmentation of $(\hat{\mathcal{F}}^\infty_t) := \sigma(v^1_\infty(s), v^2_\infty(s), M^1_\infty(s), M^2_\infty(s), W_\infty(s))_{0\le s\le t}$. From Proposition 5.1 it follows that $v^1_\infty = v^2_\infty$.
In particular, since $M^i_\infty = \int_0^t H(v^i_\infty)\, dW_\infty$ for $i = 1, 2$, we get $M^1_\infty = M^2_\infty$. Hence

$$\mu(\{(w, z) \in \mathcal{X};\ w = z\}) = 1.$$

Thus by Lemma 6.1 the sequence $(\tilde{B}_N, M_N, W)$ converges in probability to a function

$$(B_\infty, M_\infty, W) : \Omega \to L^2(0,T;L^2(D)) \times \mathcal{C}([0,T];L^2(D)) \times \mathcal{C}([0,T];U).$$

Since $(\tilde{B}_N, M_N, W)$ converges in probability, there exists a not relabeled subsequence of $(\tilde{B}_N, M_N, W)$ which convergences a.s. in Ω. Now we are in the same situation as in Section 4, but with respect to the probability space Ω instead of $\hat{\Omega}$. We repeat all arguments in Section 4, so we see that $v_\infty := B^{-1}(B_\infty)$ is a strong solution to (P). In fact, this solution is unique since the assertion in Proposition 5.1 holds true.

7 Appendix

Theorem 7.1. *Let $a : D \times \mathbb{R}^d \to \mathbb{R}^d$ and $b : \mathbb{R} \to \mathbb{R}$ as in section 2. Then the operator $A : W^{1,p}_0(D) \to W^{-1,p'}(D)$, $Au = -\mathrm{div}\ a(\cdot, \nabla(b^{-1}(u)))$ is pseudomonotone.*

Proof. Let $u_n \rightharpoonup u$ in $W^{1,p}_0(D)$ and $\limsup_{n\to\infty} \langle Au_n, u_n - u\rangle \le 0$.

Then we have

$$\begin{aligned}
\langle Au_n, u_n - u\rangle &= \int_D a(x, \nabla(b^{-1}(u_n))) \cdot \nabla(u_n - u) \\
&= \int_D \Big(a(x, (b^{-1})'(u_n)\nabla u_n) - a(x, (b^{-1})'(u_n)\nabla u) \Big) \cdot \nabla(u_n - u) \\
&+ \int_D a(x, (b^{-1})'(u_n)\nabla u) \cdot \nabla(u_n - u) \\
&= \int_D \frac{1}{(b^{-1})'(u_n)} \Big(a(x, (b^{-1})'(u_n)\nabla u_n) - a(x, (b^{-1})'(u_n)\nabla u) \Big) \cdot \\
&\cdot ((b^{-1})'(u_n)\nabla u_n - (b^{-1})'(u_n)\nabla u) + \int_D a(x, (b^{-1})'(u_n)\nabla u) \cdot \nabla(u_n - u) \\
&\geq \int_D a(x, (b^{-1})'(u_n)\nabla u) \cdot \nabla(u_n - u) \to 0.
\end{aligned}$$

Here we used that $(b^{-1})' > 0$, a is monotone and $(b^{-1})'(u_n) \to (b^{-1})'(u)$ in $L^p(D)$.
Therefore we obtain $\liminf_{n\to\infty} \langle Au_n, u_n - u\rangle \geq 0$.
By using the assumption we may conclude $\lim_{n\to\infty} \langle Au_n, u_n - u\rangle = 0$.

Now let $w \in W_0^{1,p}(D)$ and set $z = u + t(w - u)$, $t > 0$. It follows that $z \to u$ in $W_0^{1,p}(D)$ for $t \to 0^+$. We obtain:

$$\begin{aligned}
\langle Au_n - Az, u_n - z\rangle &= \int_D \Big(a(x, (b^{-1})'(u_n)\nabla u_n) - a(x, (b^{-1})'(z)\nabla z) \Big) \cdot \nabla(u_n - z) \\
&= \int_D \Big(a(x, (b^{-1})'(u_n)\nabla u_n) - a(x, (b^{-1})'(u_n)\nabla z) \Big) \cdot \nabla(u_n - z) \\
&+ \int_D \Big(a(x, (b^{-1})'(u_n)\nabla z) - a(x, (b^{-1})'(z)\nabla z) \Big) \cdot \nabla(u_n - z) \\
&\geq \int_D \Big(a(x, (b^{-1})'(u_n)\nabla z) - a(x, (b^{-1})'(z)\nabla z) \Big) \cdot \nabla(u_n - z).
\end{aligned}$$

Therefore it follows

$$\begin{aligned}
t\langle Au_n, u - w\rangle &\geq -\langle Au_n, u_n - u\rangle + t\langle Az, u - w\rangle + \langle Az, u_n - u\rangle \\
&+ \int_D \Big(a(x, (b^{-1})'(u_n)\nabla z) - a(x, (b^{-1})'(z)\nabla z) \Big) \cdot \nabla(u_n - z).
\end{aligned}$$

Now we take the limit inferior on both sides of the inequality. Since a is monotone we may conclude

$$\begin{aligned}
t \liminf_{n\to\infty} \langle Au_n, u_n - w\rangle &\geq t\langle Az, u - w\rangle + \int_D \Big(a(x, (b^{-1})'(u)\nabla z) - a(x, (b^{-1})'(z)\nabla z) \Big) \cdot \nabla(u - z) \\
&= t\langle Az, u - w\rangle + t\int_D \Big(a(x, (b^{-1})'(u)\nabla z) - a(x, (b^{-1})'(z)\nabla z) \Big) \cdot \nabla(u - w)
\end{aligned}$$

We divide by t and get

$$\liminf_{n\to\infty}\langle Au_n, u_n - w\rangle \geq \langle Az, u - w\rangle + \int_D \Big(a(x, (b^{-1})'(u)\nabla z) - a(x, (b^{-1})'(z)\nabla z) \Big) \cdot \nabla(u - w)$$

By passing to the limit $t \to 0^+$ we get:

$$\begin{aligned}\liminf_{n\to\infty}\langle Au_n, u_n - w\rangle &\geq \langle Au, u - w\rangle + \int_D \Big(a(x, (b^{-1})'(u)\nabla u) - a(x, (b^{-1})'(u)\nabla u) \Big) \cdot \nabla(u - w)\\ &= \langle Au, u - w\rangle.\end{aligned}$$

Hence, A is pseudomonotone. □

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