On Jacobsthal Binary Sequences

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Abstract

Let $\Sigma = \{0, 1\}$ be the binary alphabet, and $A = \{0, 01, 11\}$ the set of three strings $0, 01, 11$ over Σ. Let A^* denote the Kleene closure of A, $\mathbb{Z}^0$ the set of nonnegative integers, and $\mathbb{Z}^+$ the set of positive integers. A sequence in A^* is called a Jacobsthal binary sequence. Let $J(n)$ denote the set of Jacobsthal binary sequences of length n. For $k \in \mathbb{Z}^+$, $\{s_1, s_2, \ldots, s_k\} \subset \mathbb{Z}^0$, and $n - 1 \geqslant s_1 > s_2 > \ldots > s_k \geqslant 0$, let $J_1(n; s_1, s_2, \ldots, s_k)$ denote the subset $J_1(n; s_1, s_2, \ldots, s_k) = \{a_{n-1}a_{n-2} \ldots a_1 a_0 \in J(n) : a_{s_i} = 1\ (1 \leqslant i \leqslant k)\}$, of $J(n)$, and let $N_1(n; s_1, s_2, \ldots, s_k) = |J_1(n; s_1, s_2, \ldots, s_k)|$. When $k = 1$, a formula for $N_1(n; s)$ has been derived recently. In this paper we consider the general case of $N_1(n; s_1, s_2, \ldots, s_k)$, and study some other special types of Jacobsthal binary sequences. Some identities involving these numbers are also given.

Keywords. Jacobsthal numbers, combinatorial identities, combinatorial enumeration

Introduction

Let $\Sigma = \{0, 1\}$ be the binary alphabet, and $A = \{0, 01, 11\}$ the set of three strings $0, 01, 11$ over Σ. Let A^* denote the Kleene closure of A, $\mathbb{Z}^0$ the set of nonnegative integers, and $\mathbb{Z}^+$ the set of positive integers. A sequence in A^* is called a Jacobsthal binary sequence. Let $J(n)$ denote the set of Jacobsthal binary sequences of length n and let $|J(n)|$ denote the cardinality of $J(n)$.

The Jacobsthal numbers are defined by the recursion

$$J_n = J_{n-1} + 2J_{n-2}, \quad n > 2 \tag{1}$$

together with the initial values

$$J_0 = J_1 = 1. \tag{2}$$

Note that some other authors use the initial values $J_0 = 0,\ J_1 = 1$ instead. Using the initial values in (2), a known result can be stated more conveniently as

$$|J(n)| = J_n. \tag{3}$$

J_n is also called the n^{th} Jacobsthal number. For convenience, we also define

$$J_m = 0, \forall m \in \mathbb{Z},\ m < 0. \tag{4}$$

Based on (4), we state an obvious fact and a known result as a lemma for easy reference.

Lemma 1 *The recursion (1) can be extended as*

$$J_t = J_{t-1} + 2J_{t-2}, \quad t \in \mathbb{Z},\ t \neq 0.$$

The value of J_n $(n \in \mathbb{Z}^0)$ *can be computed by*

$$J_n = \frac{1}{3}(2^{n+1} + (-1)^n), \quad n \in \mathbb{Z}^0. \tag{5}$$

The Jacobsthal numbers have applications in such areas as tiling, graph matching, alternating sign matrices, etc. ([1, 2, 4, 5]).

Let

$$k \in \mathbb{Z}^+,\ \{s_1, s_2, \ldots, s_{k-1}, s_k\} \subset \mathbb{Z}^0\, ; n-1 \geqslant s_1 > s_2 > \ldots > s_k \geqslant 0. \tag{6}$$

Let $J_1(n; s_1, s_2, \ldots, s_k)$ denote the following subset of $J(n)$:

$$J_1(n; s_1, s_2, \ldots, s_k) = \{a_{n-1}a_{n-2}\ldots a_1a_0 \in J(n) : a_{s_i} = 1\ (1 \leqslant i \leqslant k)\},$$

i.e., the subset of Jacobsthal binary sequences that have the digit 1 at each of the s_i^{th} $(1 \leqslant i \leqslant k)$ positions from the right. Let $N_1(n; s_1, s_2, \ldots, s_k) = |J_1(n; s_1, s_2, \ldots, s_k)|$. R. Grimaldi[4] considers the case where $k = 1$, establishing a recursion for $N_1(n; s_1)$ and then deriving the following formula:

$$N_1(n; s) = \frac{1}{3}(2^n + (-1)^n + (-1)^{n-s}2^s) \tag{7}$$

$$= J_n - \frac{2^s}{3}(2^{n-s} + (-1)^{n-s-1}). \tag{8}$$

For the general case, finding a formula for $N_1(n; s_1, s_2, \ldots, s_k)$ by using a recursion seems extremely difficult. In this article we employ a different approach to dealing with this problem, namely, considering the following dual problem of $N_1(n; s_1, s_2, \ldots, s_k)$.

Let

$$r \in \mathbb{Z}^+,\ \{t_1, t_2, \ldots, t_{r-1}, t_r\} \subset \mathbb{Z}^0,\ n-1 \geqslant t_1 > t_2 > \ldots > t_r \geqslant 0. \tag{9}$$

Let $J_0(n; t_1, t_2, \ldots, t_r)$ denote the following subset of $J(n)$:

$$J_0(n; t_1, t_2, \ldots, t_r) = \{a_{n-1}a_{n-2}\ldots a_1a_0 \in J(n) : a_{t_i} = 0\ (1 \leqslant i \leqslant r)\},$$

i.e., the subset of Jacobsthal binary sequences that have the digit 0 at each of the t_i^{th} $(1 \leqslant i \leqslant r)$ positions from the right. Let $N_0(n; t_1, t_2, \ldots, t_r) = |J_0(n; t_1, t_2, \ldots, t_r)|$.

In the next section we present characterizations of the sets $J(n)$ and $J_0(n; t_1, t_2, \ldots, t_r)$. Based on them, some combinatorial identities involving J_n, $N_0(n; t_1, t_2, \ldots, t_r)$ and $N_1(n; s_1, s_2, \ldots, s_k)$ are derived in Section 3. From these identities, formulas for $N_0(n; t_1, t_2, \ldots, t_r)$ and $N_1(n; s_1, s_2, \ldots, s_k)$ are obtained in the last section.

1 Characterizations of the sets $J(n)$ and $J_0(n; t_1, t_2, \ldots, t_r)$

For easy reference we state a trivial fact, that is

Lemma 2 *For any* $i, j \in \mathbb{Z}^+$, $J(i)||J(j) \subseteq J(i+j)$, *where* $J(i)||J(j) = \{a||b : a \in J(i), b \in J(j)$ *and* $||$ *stands for the concatenation operation on strings.*

We now characterize the set $J(n)$. We need

Lemma 3 *Let* $l \in \mathbb{Z}^+$. *The string* α *of the 0-digit followed by* $l-1$ *1-digits is a Jacobsthal binary string of length* l.

Proof. If $l = 2m+1$ for some $m \in \mathbb{Z}^0$, the $l-1 = 2m$ 1-digits in α can be regarded as m copies of the string 11. Since both strings 11, $0 \in A$, we know $\alpha \in A$. If $l = 2m$ for some $m \in \mathbb{Z}^0$, the last $l-2 = 2m-2$ 1-digits in α can be regarded as $m-1$ copies of the string 11. Since both string 11, $01 \in A$, we know $\alpha \in A$. □

Theorem 1 *For any* $n \in \mathbb{Z}^+$, *a binary sequence of length* n *is in* $J(n)$ *if and only if it is an all-1 sequence of even length or its first 0-digit from the left is preceded by an all-1 subsequence of even length.*

Proof. Since the string $1 \notin A$ but the string $11 \in A$, the all-1 sequence of length n is in $J(n)$ if and only if n is even. Therefore, in what follows we only need to consider the case in which the sequence $a_{n-1}a_{n-2}\ldots a_1a_0$ has at least one 0-digit.

Let a_{n-i} be the first 0-digit from the left. Then

$$a_{n-1} = a_{n-2} = \ldots = a_{n-(i-1)} = 1.$$

Since the two strings $1, 10 \notin A$, in order for $a_{n-1}a_{n-2}\ldots a_1a_0$ to be in $J(n)$, the subsequence $a_{n-1}a_{n-2}\ldots a_{n-(i-1)}$ has to be formed by copies of the element $11 \in A$. This is impossible when $i-1$ is odd.

We now prove that when $i-1$ is even, the sequence $a_{n-1}a_{n-2}\ldots a_1a_0$ is in $J(n)$ by induction on the number, say u, of 0-digits in the sequence. For the case where $u = 1$, let $a_i = 0$,. By Lemma 3, the subsequence $a_ia_{i-1}\ldots a_1a_0 \in J(i+1)$. Recalling that $a_{n-1}a_{n-2}\ldots a_{i+1} \in J(n-i-1)$ we know $a_{n-1}a_{n-2}\ldots a_1a_0 \in J(n)$ by Lemma 2. This establishes the induction basis.

For the inductive step, suppose that $u > 1$ and the conclusion is true for any sequence having exactly $u-1$ 0-digits. Let a_l be the first 0-digit from the right in a sequence having u 0-digits. By Lemma 3, we know $a_la_{l-1}\ldots a_0 = 011\ldots1\ldots a_0 \in J(l+1)$. By the induction hypothesis, $a_{n-1}a_{n-2}\ldots a_{l+1} \in J(n-l-1)$. Therefore, $a_{n-1}a_{n-2}\ldots a_1a_0 \in J(n)$ by Lemma 2. This completes the induction. □

From this theorem, one can obtain the known formula (5) for $|J(n)|$.

Corollary 1

$$|J(n)| = \frac{2^{n+1} + (-1)^n}{3},$$

Proof. Let $J(n,i)$ denote the set of such Jacobsthal binary sequences that have their first 0-digit at the $(2i+1)^{st}$ position from the left, and Δ_n the set consisting of the all-1 sequence of length n when $2 \mid n$, and $\Delta_n = \emptyset$ when $2 \nmid n$. Then

$$J(n) = \Big(\bigcup_{0 \leqslant i \leqslant (n-1)/2} J(n,i) \Big) \cup \Delta_n$$

is a partition of $J(n)$. By Theorem 1, when $n = 2m$ ($m \in \mathbb{Z}^+$), we have :

$$|J(n)| = \sum_{i=0}^{m-1} 2^{2m-(2i+1)} + 1 = \frac{1}{2}\sum_{i=0}^{m-1} 4^{(m-i)} + 1 = \frac{1}{2}\sum_{i=1}^{m} 4^{i} + 1 =$$

$$= 2\sum_{i=0}^{m-1} 4^{i} + 1 = 2(\tfrac{4^m-1}{3}) + 1 = \frac{2^{n+1}+(-1)^n}{3}.$$

When $n = 2m+1$ ($m \in \mathbb{Z}^0$), we have :

$$|J(n)| = \sum_{i=0}^{m} 2^{2m+1-(2i+1)} = \sum_{i=0}^{m} 2^{2(m-i)} = \sum_{i=0}^{m} 2^{2i} =$$

$$= \sum_{i=0}^{m} 4^{i} = \frac{4^{m+1}-1}{3} = \frac{2^{n+1}+(-1)^n}{3}. \quad \square$$

By Theorem 1 we can give a characterization of the set $J_0(n; t_1, t_2, \ldots, t_r)$. Recall that the parameters satisfy (9):

$$r \in \mathbb{Z}^+,\ \{t_1, t_2, \ldots, t_{r-1}, t_r\} \subset \mathbb{Z}^0,\ n-1 > t_1 > t_2 > \ldots > t_r \geqslant 0.$$

Theorem 2 *For any $n \in \mathbb{Z}^+$, the binary sequence $a_{n-1}a_{n-2}\ldots a_1a_0$ of length n is in $J_0(n; t_1, t_2, \ldots, t_r)$ if and only if the subsequence $a_{n-1}a_{n-2}\ldots a_{t_1+1}$ is in $J(n-1-t_1)$ and $a_{t_i} = 0$ $(1 \leqslant i \leqslant r)$.*

Proof. Let a_j be the first 0-digit from the left. Then $j \geqslant t_1$. By Theorem 1, $a_{n-1}a_{n-2}\ldots a_1a_0 \in J(n)$ if and only if the entries before a_j are all 1's, i.e., $2|n-1-j$, which is the necessary and sufficient condition for $a_{n-1}a_{n-2}\ldots a_{t_1+1}$ to be in $J(n-1-t_1)$. $\square$

It is somewhat surprising that whether $a_{n-1}a_{n-2}\ldots a_1a_0 \in J_0(n; t_1, t_2, \ldots, t_r)$ or not is determined only by the subsequence $a_{n-1}a_{n-2}\ldots a_{t_1+1}$ and $a_{t_i} = 0$ $(1 \leqslant i \leqslant r)$, but is independent of the digits a_j $(0 \leqslant j \leqslant t_1 - 1,\ j \neq t_i)$.

Based on these theorems, some combinatorial identities involving J_n, $N_0(n; t_1, t_2, \ldots, t_r)$ and $N_1(n; s_1, s_2, \ldots, s_k)$ can be established, which will be presented in the next section.

2 Some Combinatorial Identities Involving J_n, $N_0(n; t_1, t_2, \ldots, t_r)$ and $N_1(n; s_1, s_2, \ldots, s_k)$

In this section some combinatorial identities involving J_n, $N_0(n; t_1, t_2, \ldots, t_r)$ and $N_1(n; s_1, s_2, \ldots, s_k)$ are proved. Applying them to obtain formulas for $N_0(n; t_1, t_2, \ldots, t_r)$ and $N_1(n; s_1, s_2, \ldots, s_k)$ will be the task of the next section.

We need a simple lemma :

Lemma 4 *For any* $n \in \mathbb{Z}^0$,

$$2^n = 3J_{n-1} + (-1)^n.$$

Proof. Recalling that $J_{-1} = 0$ (cf. (4)), we know that the statement is true when $n = 0$. When $n \in \mathbb{Z}^+$, the statement is equivalent to (5). $\square$

We can now state the following

Theorem 3

$$N_0(n; t_1, t_2, \dots, t_r) = [3J_{t_1-r} + (-1)^{t_1-r+1}]J_{n-t_1-1} \tag{10}$$
$$N_0(n; t_1, t_2, \dots, t_r) = J_{n-r} + (-1)^{n-t_1-1}J_{t_1-r} \tag{11}$$

Proof. By Theorem 2, for a sequence $a_{n-1}a_{n-2}\dots a_1a_0$ in $J_0(n; t_1, t_2, \dots, t_r)$, there are $|J(n-t_1-1)| = J_{n-t_1-1}$ many choices for the subsequences $a_{n-1}a_{n-2}\dots a_{t_1+1}$. For each of these choices, there are two choices for each of the digits a_j $(0 \leqslant j \leqslant t_1 - 1, j \neq t_2, t_3, \dots, t_r)$. Noting that $a_{t_j} = 0$ $(1 \leqslant j \leqslant r)$, we have

$$\begin{aligned} N_0(n; t_1, t_2, \dots, t_r) &= |J(n - t_1 - 1)| \cdot 2^{t_1+1-r} \\ &= J_{n-t_1-1}2^{t_1-r+1}. \end{aligned}$$

By Lemma 4,

$$2^{t_1-r+1} = 3J_{t_1-r} + (-1)^{t_1-r+1}.$$

Therefore,

$$N_0(n; t_1, t_2, \dots, t_r) = J_{n-t_1-1}[3J_{t_1-r} + (-1)^{t_1-r+1}],$$

which is (10). Similarly, we can also write

$$\begin{aligned} & N_0(n; t_1, t_2, \dots, t_r) = \\ =\ & J_{n-t_1-1}2^{t_1-r+1} \\ =\ & \frac{1}{3}[2^{n-t_1} + (-1)^{n-t_1-1}]2^{t_1-r+1} \\ =\ & \frac{1}{3}[2^{n-r+1} + (-1)^{n-t_1-1}2^{t_1-r+1}] \\ =\ & \frac{1}{3}\{3J_{n-r} + (-1)^{n-r+1} + (-1)^{n-t_1-1}[3J_{t_1-r} + (-1)^{t_1-r+1}]\} \\ =\ & J_{n-r} + (-1)^{n-t_1-1}J_{t_1-r}, \end{aligned}$$

which proves (11). $\square$

From this theorem, an identity can be immediately derived.

Corollary 2 *We have the identity*

$$[3J_{t_1-r} + (-1)^{t_1-r+1}]J_{n-t_1-1} = J_{n-r} + (-1)^{n-t_1-1}J_{t_1-r}.$$

This identity can also be checked by using (5).

Let us look at the cases $r = 1$ and $r = 2$.

Corollary 3 *If $n-1 \geqslant u \geqslant 0$, then*

$$N_0(n;u) = [3J_{u-1} + (-1)^u]J_{n-u-1} \quad (12)$$
$$N_0(n;u) = J_{n-1} + (-1)^{n-u-1}J_{u-1} \quad (13)$$

Example 1 *From (13) and $J_0 = J_1 = 1, J_2 = 3$, we have*

$$\begin{aligned} N_0(1;0) &= J_0 + (-1)^0 J_{-1} = 1, \\ N_0(2;0) &= J_1 + (-1)^1 J_{-1} = 1, \\ N_0(2;1) &= J_1 + (-1)^0 J_0 = 2, \\ N_0(3;0) &= J_2 + (-1)^2 J_{-1} = 3, \\ N_0(3;1) &= J_2 + (-1)^1 J_0 = 2, \\ N_0(3;2) &= J_2 + (-1)^0 J_1 = 4. \end{aligned}$$

The corresponding subsets of $J(n)$ are

$J_0(1;0) = \{0\}, J_0(2;0) = \{00\}, J_0(2;1) = \{00, 01\}$.
$J_0(3;0) = \{000, 010, 110\}, J_0(3;1) = \{000, 001\}, J_0(3;2) = \{000, 001, 010, 011\}$.

Corollary 4 *If $n-1 \geqslant u \geqslant 0$, then*

$$[3J_{u-1} + (-1)^u]J_{n-u-1} = J_{n-1} + (-1)^{n-u-1}J_{u-1} \ .$$

For $N_1(n; s_1, s_2, \ldots, s_k)$, we have

Theorem 4 *Suppose that $s_1, s_2, \ldots, s_k$ satisfy (6). Then* $N_1(n; s_1, s_2, \ldots, s_k) =$

$$J_n + \textstyle\sum_{1\leqslant r\leqslant k}(-1)^r \sum_{1\leqslant i\leqslant k-r+1} \binom{k-i}{r-1}[J_{n-r} + (-1)^{n-s_i-1}J_{s_i-r}].$$

Proof. First of all, for any $1 \leqslant r \leqslant k$, by (11) we have :

$$\textstyle\sum_{1\leqslant i_1<i_2<\ldots<i_r\leqslant k} N_0(n; s_{i_1}, s_{i_2}, \ldots, s_{i_r}) =$$

$$\textstyle\sum_{1\leqslant i_1<i_2<\ldots<i_r\leqslant k} [J_{n-r} + (-1)^{n-s_{i_1}-1}J_{s_{i_1}-r}].$$

Since $1 \leqslant i_1 < i_2 < \ldots < i_r \leqslant k$, the index i_1 must satisfy $1 \leqslant i_1 \leqslant k-r+1$. After i_1 has been chosen from this range, there are $\binom{k-i_1}{r-1}$ ways of choosing $i_2, \ldots, i_r$. Since the summands $J_{n-r} + (-1)^{n-s_{i_1}-1}J_{s_{i_1}-r}$ do not depend on the values of $i_2, \ldots, i_r$, we have :

$$\textstyle\sum_{1\leqslant i_1<i_2<\ldots<i_r\leqslant k} [J_{n-r} + (-1)^{n-s_{i_1}-1}J_{s_{i_1}-r}] =$$

$$\textstyle\sum_{1\leqslant i_1\leqslant k-r+1} \binom{k-i_1}{r-1} [J_{n-r} + (-1)^{n-s_{i_1}-1}J_{s_{i_1}-r}] \ .$$

Further, using i to substitute for i_1 in the summation on the right hand side, yields :

$$\sum_{1\leqslant i_1<i_2<\ldots<i_r\leqslant k} N_0(n;s_{i_1},s_{i_2},\ldots,s_{i_r}) \ =$$

$$\sum_{1\leqslant i\leqslant k-r+1}\binom{k-i}{r-1}\,[J_{n-r}+(-1)^{n-s_i-1}J_{s_i-r}]\ .$$

By the inclusion-exclusion principle, $N_1(n;s_1,s_2,\ldots,s_k)\ =$

$$J_n+\sum_{1\leqslant r\leqslant k}(-1)^r\sum_{1\leqslant i_1<i_2<\ldots<i_r\leqslant k} N_0(n;s_{i_1},s_{i_2},\ldots,s_{i_r})\ =$$

$$J_n+\sum_{1\leqslant r\leqslant k}(-1)^r\sum_{1\leqslant i\leqslant k-r+1}\binom{k-i}{r-1}\,[J_{n-r}+(-1)^{n-s_i-1}J_{s_i-r}\ ,$$

which proves (4) . $\square$

Similarly, using (10) instead of (11) yields the following :

Theorem 5 *Suppose that* $s_1,s_2,\ldots,s_k$ *satisfy (6). Then* $N_1(n;s_1,s_2,\ldots,s_k)\ =$

$$J_n+\sum_{1\leqslant r\leqslant k}(-1)^r\sum_{1\leqslant i\leqslant k-r+1}\binom{k-i}{r-1}\,[3J_{s_i-r}+(-1)^{s_i-r+1}]J_{n-s_i-1}.$$

Let us look at the cases for $k=1,\ 2.$

Corollary 5 *For any* $n\in\mathbb{Z}^+$ *and* $n-1\geqslant u\geqslant 0$,

$$\begin{aligned}N_1(n;u) &= 2J_{n-2}+(-1)^{n-u}J_{u-1}\\ N_1(n;u) &= J_n-[3J_{u-1}+(-1)^u]J_{n-u-1}.\end{aligned}$$

Proof. By Theorem 4 and Lemma 1,

$$\begin{aligned}N_1(n;u) &= J_n+(-1)^1\binom{1-1}{1-1}[J_{n-1}+(-1)^{n-u-1}J_{u-1}\\ &= J_n-J_{n-1}+(-1)^{n-u}J_{u-1}\\ &= 2J_{n-2}+(-1)^{n-u}J_{u-1}\ .\end{aligned}$$

And by Theorem 5 we obtain :

$$\begin{aligned}N_1(n;u) &= J_n+(-1)^1\binom{1-1}{1-1}\,[3J_{u-1}+(-1)^uJ_{n-u-1}]\\ &= J_n-[3J_{u-1}+(-1)^uJ_{n-u-1}].\quad\square\end{aligned}$$

Example 2 *By Corollary 5, we have :*

$N_1(1;0)=2J_{-1}+J_{-1}=0,\quad N_1(2;0)=2J_0+J_{-1}=2,\quad N_1(2;1)=2J_0-J_0=1,$

$N_1(3;0)=2J_1-J_{-1}=2,\quad N_1(3;1)=2J_1+J_0=3,\quad N_1(3;2)=2J_1-J_1=1.$

The corresponding subsets of $J(n)$ *are*

$J_1(1;0)=\emptyset,\ J_1(2;0)=\{01,11\},\ J_1(2;1)=\{11\},$

$J_1(3;0)=\{001,011\},\ J_1(3;1)=\{010,011,110\},\ J_1(3;2)=\{110\}.$

Example 3 *Applying Corollary 5, we have*

$$N_1(1;0) = J_1 - [3J_{-1}+1]J_0 = 1-1 = 0.$$
$$N_1(2;0) = J_2 - [3J_{-1}+1]J_1 = 3-1 = 2.$$
$$N_1(2;1) = J_2 - [3J_0-1]J_0 = 3-2 = 1.$$
$$N_1(3;0) = J_3 - [3J_{-1}+1]J_2 = 5-3 = 2.$$
$$N_1(3;1) = J_3 - [3J_0-1]J_1 = 5-2 = 3.$$
$$N_1(3;2) = J_3 - [3J_1+1]J_0 = 5-4 = 1.$$

The corresponding subsets of $J(n)$ *have been shown in Example 2.*

Now let us turn to the case of $k=2$. In this case, $n>1$.

Corollary 6 *For any* $n \in \mathbb{Z}^+$, $n \geqslant 2$, *and* $n-1 \geqslant u > v \geqslant 0$, *we have :*

$$N_1(n;u,v) = 2[J_{n-2} - J_{n-3}] + (-1)^{n-u}[J_{u-1} - J_{u-2}] + (-1)^{n-v}J_{v-1}. \tag{14}$$

For any $n \in \mathbb{Z}^+$, $n \geqslant 3$, $n-1 \geqslant u > v \geqslant 0$, $u \geqslant 2$, *we have :*

$$N_1(n;u,v) = 4J_{n-4} + (-1)^{n-u}2J_{u-3} + (-1)^{n-v}J_{v-1}. \tag{15}$$

Proof. By Theorem 4, $N_1(n;s_1,s_2) =$

$$J_n + (-1)^1 \sum_{1\leqslant i\leqslant 2} \tbinom{2-i}{1-1}[J_{n-1} + (-1)^{n-s_i-1}J_{s_i-1}] + $$
$$+ \tbinom{2-1}{2-1}[J_{n-2} + (-1)^{n-s_1-1}J_{s_1-2}] =$$

$$J_n - [J_{n-1} + (-1)^{n-s_1-1}J_{s_1-1} + J_{n-1} + (-1)^{n-s_2-1}J_{s_2-1}] +$$
$$+ [J_{n-2} + (-1)^{n-s_1-1}J_{s_1-2}] =$$

$$J_n - 2J_{n-1} + J_{n-2} + (-1)^{n-s_1}J_{s_1-1} + (-1)^{n-s_2}J_{s_2-1} +$$
$$+(-1)^{n-s_1-1}J_{s_1-2} =$$

$$2[J_{n-2} - J_{n-3}] + (-1)^{n-s_1}[J_{s_1-1} - J_{s_1-2}] + (-1)^{n-s_2}J_{s_2-1}.$$

Substituting u, v for s_1, s_2, respectively, gives (14).

When $n \geqslant 3$, and $s_1 \geqslant 2$, by Lemma 1 we have :

$$J_{n-2} - J_{n-3} = 2J_{n-4}, \quad J_{s_1-1} - J_{s_1-2} = 2J_{s_1-3}.$$

So ,

$$\begin{aligned} N_1(n;s_1,s_2) &= 2[J_{n-2} - J_{n-3}] + (-1)^{n-s_1}[J_{s_1-1} - J_{s_1-2}] + (-1)^{n-s_2}J_{s_2-1} \\ &= 4J_{n-4} + (-1)^{n-s_1}2J_{s_1-3} + (-1)^{n-s_2}J_{s_2-1}. \end{aligned}$$

Substituting u, v for s_1, s_2, respectively, gives (15). □

The identities in this section can be used to give formulas for $N_0(n;t_1,t_2,\ldots,t_r)$ and $N_1(n;s_1,s_2,\ldots,s_k)$, which will be presented in the next section.

3 Formulas for $N_0(n; t_1, t_2, \ldots, t_r)$ and $N_1(n; s_1, s_2, \ldots, s_k)$

For $N_0(n; t_1, t_2, \ldots, t_r)$, we have:

Theorem 6 *The following holds :*

$$N_0(n; t_1, t_2, \ldots, t_r) = (\frac{1}{3})2^{t_1+1-r}[2^{n-t_1} + (-1)^{n-t_1-1}] \tag{16}$$

Proof. From the proof of Theorem 3 and equality (5), we have

$$\begin{aligned} N_0(n; t_1, t_2, \ldots, t_r) &= J_{n-1-t_1} \cdot 2^{t_1+1-r} \\ &= \frac{1}{3} 2^{t_1+1-r}[2^{n-t_1} + (-1)^{n-t_1-1}] . \quad \square \end{aligned}$$

Note that $N_0(n; t_1, t_2, \ldots, t_r)$ only depends on the parameters n, t_1 and r, and is independent of the values of the parameters $t_2, \ldots, t_r$.

Theorems 3 and 4 provide an explicit formulas for $N_1(n; s_1, s_2, \ldots, s_k)$, as shown in the following theorem. Its proof is obvious and will be omitted.

Theorem 7 *Suppose that* $s_1, s_2, \ldots, s_k$ *satisfy (6). Then* $N_1(n; s_1, s_2, \ldots, s_k) =$

$$(\tfrac{1}{3})(2^{n+1} + (-1)^n) +$$
$$+ (\tfrac{1}{3}) \sum_{1 \leqslant r \leqslant k} (-1)^r \sum_{1 \leqslant i \leqslant k-r+1} \binom{k-i}{r-1} 2^{s_i-r+1}(2^{n-s_i} + (-1)^{n-s_i-1}).$$

When $k = 1$, we have :

Corollary 7

$$N_1(n; s) = \frac{1}{3}\{2^{n+1} - 2^s[2^{n-s} + (-1)^{n-s-1}] + (-1)^n\}. \tag{17}$$

Example 4 *By (17), the first several values of* $N_1(n; s)$ *can be computed as follows.*

$$\begin{aligned} N_1(1; 0) &= \frac{1}{3}\{2^2 - 2^0[2^2 + (-1)^1] + (-1)^1\} = 0, \\ N_1(2; 0) &= \frac{1}{3}\{2^3 - 2^0[2^2 + (-1)^1] + (-1)^2\} = 2, \\ N_1(2; 1) &= \frac{1}{3}\{2^3 - 2^1[2^1 + (-1)^0] + (-1)^2\} = 1, \\ N_1(3; 0) &= \frac{1}{4}\{2^4 - 2^0[2^3 + (-1)^2] + (-1)^3\} = 2, \\ N_1(3; 1) &= \frac{1}{4}\{2^4 - 2^1[2^2 + (-1)^1] + (-1)^3\} = 3, \\ N_1(3; 2) &= \frac{1}{4}\{2^4 - 2^2[2^1 + (-1)^0] + (-1)^3\} = 1. \end{aligned}$$

The corresponding subsets of $J(n)$ *have been shown in Example 2.*

When $k = 2$, we have :

Corollary 8 *For any* $n \geqslant 2$ *and* $n - 1 \geqslant u > v \geqslant 0$, *we have :*

$$N_1(n; u, v) = (\frac{1}{3}) [2^{n-1} + (-1)^{n-u}2^{u-1} + (-1)^{n-v}2^v + (-1)^n].$$

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