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On $C^{1,\frac{1}{2}}$-regularity of $\mathcal{H}$-surfaces with a free boundary

by

Frank Müller

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Abstract

We consider stationary surfaces of prescribed mean curvature in $\mathbb{R}^3$ – shortly called $\mathcal{H}$-surfaces – with part of their boundary varying on a smooth support manifold S with non-empty boundary. We allow that the $\mathcal{H}$-surface meets the support manifold non-perpendicularly and presume the $\mathcal{H}$-surface to be continuous up to the boundary. Then we show: If S belongs to C^2 resp. $C^{2,\mu}$, then the $\mathcal{H}$-surface belongs to $C^{1,\alpha}$ for any $\alpha \in (0, \frac{1}{2})$ resp. $C^{1,\frac{1}{2}}$ up to the boundary. The latter conclusion is optimal by an example due to S. Hildebrandt and J.C.C. Nitsche. Our result extends a known theorem for the special case of minimal surfaces. In addition, we present asymptotic expansions at boundary branch points.

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Let S be a differentiable, two-dimensional manifold in $\mathbb{R}^3$ with boundary ∂S. Writing

$$B^+ := \{w = (u, v) = u + iv \,:\, |w| < 1,\ v > 0\}, \quad I := (-1, 1) \subset \partial B^+$$

for the upper unit half-disc in $\mathbb{R}^2 \simeq \mathbb{C}$ and the straight part of its boundary, we consider *surfaces of prescribed mean curvature* or shortly *$\mathcal{H}$-surfaces* on B^+, i.e. solutions of the problem

$$\begin{aligned} &\mathbf{x} \in C^2(B^+, \mathbb{R}^3) \cap C^0(\overline{B^+}, \mathbb{R}^3) \cap H_2^1(B^+, \mathbb{R}^3), \\ &\Delta \mathbf{x} = 2\mathcal{H}(\mathbf{x}) \mathbf{x}_u \wedge \mathbf{x}_v \quad \text{in } B^+, \\ &|\mathbf{x}_u| = |\mathbf{x}_v|, \quad \langle \mathbf{x}_u, \mathbf{x}_v \rangle = 0 \quad \text{in } B^+, \end{aligned} \tag{1}$$

which satisfy the *free boundary condition*

$$\mathbf{x}(I) \subset S \cup \partial S. \tag{2}$$

Here $H_2^1(B^+, \mathbb{R}^3)$ denotes the Sobolev-space of measurable mappings $\mathbf{x} : B^+ \to \mathbb{R}^3$, which are quadratically integrable together with their first derivatives. In addition, $\Delta = \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}$ stands for the Laplace operator in $\mathbb{R}^2$ and $\mathbf{y} \wedge \mathbf{z}$, $\langle \mathbf{y}, \mathbf{z} \rangle$ denote the cross-product and the scalar product in $\mathbb{R}^3$, respectively; the latter notation will be used for vectors in $\mathbb{C}^3$, too. Finally, $\mathcal{H} \in C^0(\mathbb{R}^3, \mathbb{R})$ is a precribed function. In (1), the system in the second line is called *Rellich's system* and the third line contains the *conformality relations*.

As is well-known, the restriction $\mathbf{x}|_{\mathcal{R}}$ of a solution of (1) to the set

$$\mathcal{R} := \{w \in B^+ \ : \ \nabla \mathbf{x}(w) := (\mathbf{x}_u(w), \mathbf{x}_v(w)) \neq \mathbf{0}\}$$

of *regular points* describes a surface with mean curvature $H = \mathcal{H} \circ \mathbf{x}$. We emphasize that singular points with $\nabla \mathbf{x}(w) = \mathbf{0}$, so-called *branch points*, are specifically allowed. This is natural from the viewpoint of the calculus of variations: If $\mathbf{Q} \in C^1(\mathbb{R}^3, \mathbb{R}^3)$ is a vector field with $\operatorname{div} \mathbf{Q} = 2\mathcal{H}$, then solutions of (1) appear as stationary points of the functional

$$E_{\mathbf{Q}}(\mathbf{y}) := \int\limits_{B^+} \Big\{ \frac{1}{2} |\nabla \mathbf{y}|^2 + \langle \mathbf{Q}(\mathbf{y}), \mathbf{y}_u \wedge \mathbf{y}_v \rangle \Big\} \, du \, dv, \tag{3}$$

where so-called *inner* and *outer variations* $\mathbf{y}$ of $\mathbf{x}$ are allowed. Roughly speaking, inner variation means a perturbation in the parameters (u, v) and outer variations are perturbations in the space that retain the boundary condition (2); see [DHT] Section 1.4 for the exact definitions in the minimal surface case $\mathbf{Q} \equiv \mathbf{0}$. For our purposes, it suffices to give the exact definition of outer variations:

Definition 1. *Let $\mathbf{x} \in C^0(\overline{B^+}, \mathbb{R}^3) \cap H_2^1(B^+, \mathbb{R}^3)$ fulfill the boundary condition (2). A perturbation $\mathbf{x}^{(\varepsilon)}(w) := \mathbf{x}(w) + \varepsilon \boldsymbol{\phi}(w, \varepsilon)$, $0 \le \varepsilon \ll 1$, is called* outer variation of $\mathbf{x}$, *if $\boldsymbol{\phi}(\cdot, \varepsilon)$ belongs to*

$$\mathcal{A}_{\mathbf{x}} := \left\{ \mathbf{y} \in H_2^1(B^+, \mathbb{R}^3) \ : \ \begin{array}{l} \mathbf{y} = \mathbf{x} \ \text{on } \partial B^+ \setminus I \\ \mathbf{y}(w) \in S \ \text{for a.a. } w \in I \end{array} \right\}$$

for any ε, if the family of Dirichlet's integrals

$$D\big(\boldsymbol{\phi}(\cdot, \varepsilon)\big) := \int\limits_{B^+} \Big(|\boldsymbol{\phi}_u(w, \varepsilon)|^2 + |\boldsymbol{\phi}_v(w, \varepsilon)|^2 \Big) \, du \, dv, \quad 0 \le \varepsilon \ll 1,$$

is uniformly bounded in ε, and if $\boldsymbol{\phi}(\cdot, \varepsilon) \to \boldsymbol{\phi}(\cdot, 0) \in H_2^1(B^+, \mathbb{R}^3)$ $(\varepsilon \to 0+)$ holds true a.e. on B^+. The function $\boldsymbol{\phi}_0 := \boldsymbol{\phi}(\cdot, 0)$ is to be termed direction of the variation.

Definition 2. *A solution $\mathbf{x} : \overline{B^+} \to \mathbb{R}^3$ of (1)–(2) is called* stationary free $\mathcal{H}$-surface, *if we have*

$$\delta E_{\mathbf{Q}}(\mathbf{x}, \boldsymbol{\phi}_0) := \lim_{\varepsilon \to 0+} \frac{1}{\varepsilon} \big[E_{\mathbf{Q}}(\mathbf{x}^{(\varepsilon)}) - E_{\mathbf{Q}}(\mathbf{x}) \big] \ge 0$$

for any outer variation $\mathbf{x}^{(\varepsilon)} = \mathbf{x} + \varepsilon \boldsymbol{\phi}(\cdot, \varepsilon)$, $0 \le \varepsilon \ll 1$. The quantity $\delta E_{\mathbf{Q}}(\mathbf{x}, \boldsymbol{\phi}_0)$ is called the first variation of $E_{\mathbf{Q}}$ at $\mathbf{x}$ in the direction $\boldsymbol{\phi}_0$.

Now we are able to formulate our main result:

Theorem 1. *Let $S \subset \mathbb{R}^3$ be a differentiable two-manifold and assume a vectorfield $\mathbf{Q} \in C^1(\mathbb{R}^3, \mathbb{R}^3)$ to be given such that*

$$|\langle \mathbf{Q}, \mathbf{n} \rangle| < 1 \quad \text{on } S \cup \partial S \tag{4}$$

is satisfied; here $\mathbf{n} : S \cup \partial S \to \mathbb{R}^3$ denotes a unit normal field on S which we locally extend continuously to ∂S. In addition, let $\mathbf{x} \in C^2(B^+, \mathbb{R}^3) \cap C^0(\overline{B^+}, \mathbb{R}^3) \cap H_2^1(B^+, \mathbb{R}^3)$ be a stationary free $\mathcal{H}$-surface with $\mathcal{H} := \frac{1}{2} \operatorname{div} \mathbf{Q}$.

(i) If $S \in C^2$, then we have $\mathbf{x} \in C^{1,\alpha}(B^+ \cup I, \mathbb{R}^3)$ for any $\alpha \in (0, \frac{1}{2})$.

(ii) If $S \in C^{2,\beta}$ and $\mathbf{Q} \in C^{1,\beta}(\mathbb{R}^3, \mathbb{R}^3)$ for some $\beta \in (0,1)$, then we have $\mathbf{x} \in C^{1,\frac{1}{2}}(B^+ \cup I, \mathbb{R}^3)$.

Remark 1. *For minimal surfaces, i.e. the special case $\mathbf{Q} \equiv \mathbf{0}$, the result of Theorem 1 is due to R. Ye [Y]. Under higher regularity assumptions on S - namely $S \in C^3$ in case (i), $S \in C^4$ in case (ii) - these results for minimal surfaces were already proved by S. Hildebrandt and J.C.C. Nitsche [HN1], [HN2]. In [HN2] the authors present an example showing the optimality of the regularity proved in Theorem 1 (ii).*

Remark 2. *In the minimal surface case, the assumption $\mathbf{x} \in C^0(\overline{B^+}, \mathbb{R}^3)$ in Theorem 1 becomes redundant provided S satisfies an additional uniformity condition. This is the famous continuity result for stationary minimal surfaces up to the free boundary, which is due to M. Grüter, S. Hildebrandt, J.C.C. Nitsche [GHN1]; see also G. Dziuk [Dz] regarding an analogue result for support surfaces without boundary. Concerning $\mathcal{H}$-surfaces, it is an open question whether stationarity implies continuity up to the boundary. However, there is an affirmative answer in the special case of vector-fields $\mathbf{Q}$ satisfying*

$$\langle \mathbf{Q}, \mathbf{n} \rangle = 0 \quad \text{on } S \cup \partial S;$$

see [GHN2] for support surfaces without boundary, in [M2] the case of support surfaces with boundary is shortly treated. In addition, minimality – instead of the weaker assumption of stationarity – implies continuity up to the boundary under very mild assumptions on S and a smallness condition for $\mathbf{Q}$; see [DHT] Section 2.5 or [M3] Section 1.3.

Remark 3. *In the general case $\langle \mathbf{Q}, \mathbf{n} \rangle \not\equiv 0$ on $S \cup \partial S$ the only results for stationary $\mathcal{H}$-surfaces known to the author are addressed to the case of support surfaces with empty boundary $\partial S = \emptyset$, see [HJ], [Ha], [M4].*

Our second theorem is concerned with boundary branch points:

Theorem 2. *Let the assumptions of Theorem 1 (i) be satisfied and let $w_0 \in I$ be a branch point of the stationary free $\mathcal{H}$-surface $\mathbf{x}$. If $\mathbf{x} : \overline{B^+} \to \mathbb{R}^3$ is non-constant, then there exist an integer $m \geq 1$ and a vector $\mathbf{a} \in \mathbb{C}^3 \setminus \{\mathbf{0}\}$ with $\langle \mathbf{a}, \mathbf{a} \rangle = 0$, such that we have the representation*

$$\mathbf{x}_w(w) = \boldsymbol{a}(w - w_0)^m + o(|w - w_0|^m) \quad \text{as } w \to w_0. \tag{5}$$

Remark 4. *The proof of Theorem 2 can be found at the end of the paper; for branch points $w_0 \in I$ with $\mathbf{x}(w_0) \in S$ the asymptotic expansion (5) has been already proved in [M4] Theorem 1.13. The usual direct consequences as finiteness of boundary branch points in $\overline{B^+} \cap B_r(0)$ for any $r \in (0,1)$ and continuity of the surface normal of $\mathbf{x}$ up to the branch points follow; see e.g. [M4] Remarks 5.1 and 5.2.*

Starting with the proof of Theorem 1 (i) and (ii), it suffices to show that for any $w_0 \in I$ there exists some $\delta > 0$ such that $\mathbf{x} \in C^{1,\mu}(\overline{B_\delta^+(w_0)}, \mathbb{R}^3)$ with $\mu \in (0, \frac{1}{2})$ or $\mu = \frac{1}{2}$, respectively. Here we abbreviated

$$B_\delta(w_0) := \{w = u + iv \in \mathbb{C} \; : \; |w - w_0| < \delta\},$$

$$B_\delta^+(w_0) := \{w = u + iv \in B_\delta(w_0) \; : \; v > 0\}.$$

Since this result is included in Theorem 1.3 of [M4] for $w_0 \in I$ with $\mathbf{x}_0 := \mathbf{x}(w_0) \in S$, we may assume $\mathbf{x}_0 \in \partial S$. We localize around $\mathbf{x}_0$ which is possible according to the assumption $\mathbf{x} \in C^0(\overline{B^+}, \mathbb{R}^3)$. After a suitable rotation and translation we can presume $\mathbf{x}_0 = \mathbf{0}$ as well as the existence of some neighbourhood $\mathcal{U} = \mathcal{U}(\mathbf{x}_0) \subset \mathbb{R}^3$ and functions $\gamma \in C^2([-r,r])$, $\psi \in C^2(\overline{B_r(0)})$, $r > 0$, with

$$\gamma(0) = \frac{d}{ds}\gamma(0) = 0, \quad \psi(0) = \nabla\psi(0) = 0, \tag{6}$$

such that we have the local representations

$$\begin{aligned} S \cap \mathcal{U} &= \big\{\mathbf{p} = (p^1, p^2, p^3) \in \Omega \times \mathbb{R} \,:\, p^3 > \psi(p^1, p^2)\big\}, \\ \partial S \cap \mathcal{U} &= \big\{\mathbf{p} = (p^1, p^2, p^3) \in \Gamma \times \mathbb{R} \,:\, p^3 = \psi(p^1, p^2)\big\}, \end{aligned} \tag{7}$$

where we abbreviated

$$\begin{aligned} \Omega &:= \big\{(p^1, p^2) \in B_r(0) \,:\, p^2 > \gamma(p^1)\big\}, \\ \Gamma &:= \big\{(p^1, p^2) \in B_r(0) \,:\, p^2 = \gamma(p^1)\big\}. \end{aligned} \tag{8}$$

Now choose $\delta > 0$ with $|\mathbf{x}(w)| < r$ for all $w \in \overline{B^+_\delta(w_0)}$. Since the system (1) is conformally invariant, we may reparametrize $\mathbf{x}|_{\overline{B^+_\delta(w_0)}}$ over $\overline{B^+}$ without renaming and obtain

$$\mathbf{x}(\overline{B^+}) \subset \mathcal{B}_r := \big\{\mathbf{p} \in \mathbb{R}^3 \,:\, |\mathbf{p}| < r\big\}, \qquad \mathbf{x}(0) = \mathbf{0}. \tag{9}$$

In the following, we will repeatedly scale $r > 0$ down – sometimes without further command – always assuming (9) to be satisfied.

Next we define

$$q = q(\mathbf{p}) := Q^3(\mathbf{p}) - \psi_{p^1}(p^1, p^2) Q^1(\mathbf{p}) - \psi_{p^2}(p^1, p^2) Q^2(\mathbf{p}), \tag{10}$$

where Q^1, Q^2, Q^3 are the components of $\mathbf{Q}$. Note that the smallness condition (4) and the normalization (6) imply $q \in C^1(\overline{\mathcal{B}_r})$ as well as

$$|q(\mathbf{p})| \le q_0 < 1 \quad \text{for all } \mathbf{p} \in \overline{\mathcal{B}_r} \tag{11}$$

for sufficiently small $r > 0$; here $q_0 \in (0,1)$ denotes some suitable constant.

Writing $\dot\gamma := \frac{d}{ds}\gamma$, we set

$$\begin{aligned} z^1 &:= -i\psi_{p^1} x^1_w - i\psi_{p^2} x^2_w + i x^3_w, \\ z^2 &:= (1 - iq\dot\gamma) x^1_w + (\dot\gamma + iq) x^2_w + (\psi_{p^1} + \psi_{p^2}\dot\gamma) x^3_w \quad \text{on } B^+. \end{aligned} \tag{12}$$

Here we abbreviated $\psi_{p^j} = \psi_{p^j}(x^1, x^2)$, $\gamma = \gamma(x^1)$, and $q = q(\mathbf{x})$, and we used one of the Wirtinger derivatives $x^j_w = \frac{\partial x^j}{\partial w}$ defined by the operators

$$\frac{\partial}{\partial w} := \frac{1}{2}\Big(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}\Big), \quad \frac{\partial}{\partial \overline{w}} := \frac{1}{2}\Big(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\Big).$$

As a first important observation we infer the following

Proposition 1. *The mapping $\mathbf{z} := (z^1, z^2) : B^+ \to \mathbb{R}^3$ belongs to $C^1(B^+, \mathbb{C}^2) \cap L_2(B^+, \mathbb{C}^2)$ and satisfies the weak boundary condition*

$$\liminf_{\varrho \to 0} \Big| \int_{I_\varrho} \big\langle \boldsymbol{\lambda}(w), Im\, \mathbf{z}(w) \big\rangle \, du \Big| = 0 \quad \textit{for all } \boldsymbol{\lambda} \in C^1_c(B^+ \cup I, \mathbb{R}^2), \tag{13}$$

where we set $I_\varrho := \{w = u + iv \in B^+ \,:\, v = \varrho\}$ for $\varrho > 0$.

Proof. The claimed regularity of $\mathbf{z}$ is obvious by definition. In order to prove (13), we set $\eta(s) := \psi(s,\gamma(s))$ and $\mathbf{t}(s) := (1,\dot\gamma(s),\dot\eta(s))$, $s\in(-r,r)$. Then $\mathbf{t}(s)$ is tangential to ∂S at the point $(s,\gamma(s),\eta(s))$. If we choose $\alpha \in C_c^1(B^+\cup I)$ arbitrarily, the stationarity of $\mathbf{x}$ yields

$$\lim_{\varrho\to 0+}\int\limits_{I_\varrho} \alpha\big\langle \mathbf{t}(x^1),\mathbf{x}_v+\mathbf{Q}(\mathbf{x})\wedge\mathbf{x}_u\big\rangle\,du = 0; \tag{14}$$

this can be proved by combining the flow argument in [DHT] pp. 32–33 with [M1] Lemma 3. Now we set $\zeta := \langle \mathbf{t}(x^1),\mathbf{x}_v+\mathbf{Q}(\mathbf{x})\wedge\mathbf{x}_u\rangle$ and claim

$$2\,\mathrm{Im}\,z^2 = -\zeta + (Q^2-\dot\gamma Q^1)(x_u^3-\psi_{p^1}x_u^1-\psi_{p^2}x_u^2) \quad \text{on } B^+, \tag{15}$$

where we again abbreviated $Q^j = Q^j(\mathbf{x})$, etc. Indeed, we compute

$$\begin{aligned}\zeta &= x_v^1+Q^2x_u^3-Q^3x_u^2+\dot\gamma(x_v^2+Q^3x_u^1-Q^1x_u^3)+\dot\eta(x_v^3+Q^1x_u^2-Q^2x_u^1)\\ &= x_v^1+\dot\gamma x_v^2-(Q^3-\psi_{p^1}Q^1-\psi_{p^2}Q^2)(x_u^2-\dot\gamma x_u^1)+(\psi_{p^1}+\psi_{p^2}\dot\gamma)x_v^3\\ &\quad +(Q^2-\dot\gamma Q^1)(x_u^3-\psi_{p^1}x_u^1-\psi_{p^2}x_u^2)\quad \text{on } B^+,\end{aligned}$$

having $\dot\eta = \psi_{p^1}+\psi_{p^2}\dot\gamma$ in mind. Hence, the definition (12) of z^2 yields (15).

Next we note the inequality

$$\int\limits_{I_\varrho}[x^3-\psi(x^1,x^2)]^2\,du \le c\varrho\int\limits_{B^+}|\nabla\mathbf{x}|^2\,du\,dv \le c\varrho, \quad \delta\in(0,1), \tag{16}$$

with some constant $c>0$. This is an easy consequence of the boundary condition $x^3=\psi(x^1,x^2)$ on I and the boundedness of $|\nabla\psi|$.

Now let $\boldsymbol{\lambda} = (\lambda_1,\lambda_2)\in C_c^1(B^+\cup I,\mathbb{R}^2)$ be chosen arbitrarily. Then we estimate

$$\begin{aligned}&\liminf_{\varrho\to 0}\Big|\int\limits_{I_\varrho}\big\langle\boldsymbol{\lambda}(w),\mathrm{Im}\,\mathbf{z}(w)\big\rangle\,du\Big|\\ &= \liminf_{\varrho\to 0}\Big|\int\limits_{I_\varrho}\big(\lambda_1\,\mathrm{Im}\,z^1+\lambda_2\,\mathrm{Im}\,z^2\big)\,du\Big|^2\\ &\overset{(14),(15)}{=} \liminf_{\varrho\to 0}\frac14\Big|\int\limits_{I_\varrho}\big[\lambda_1+\lambda_2(Q^2-\dot\gamma Q^1)\big]\big[x_u^3-\psi_{p^1}x_u^1-\psi_{p^2}x_u^2\big]\,du\Big|^2\\ &= \liminf_{\varrho\to 0}\frac14\Big|\int\limits_{I_\varrho}\big[x^3-\psi(x^1,x^2)\big]\frac{\partial}{\partial u}\big[\lambda_1+\lambda_2(Q^2-\dot\gamma Q^1)\big]\,du\Big|^2\\ &\le \liminf_{\varrho\to 0}\frac14\int\limits_{I_\varrho}\big[x^3-\psi(x^1,x^2)\big]^2\,du\cdot\int\limits_{I_\varrho}\Big\{\frac{\partial}{\partial u}\big[\lambda_1+\lambda_2(Q^2-\dot\gamma Q^1)\big]\Big\}^2\,du\\ &\overset{(16)}{\le} \liminf_{\varrho\to 0} c\varrho\Big(1+\int\limits_{I_\varrho}|\nabla\mathbf{x}|^2\,du\Big).\end{aligned}$$

with an adjusted constant $c>0$. Using $\mathbf{x}\in H_2^1(B^+,\mathbb{R}^3)$, one can easily prove that the right hand side of this inequality vanishes (see e.g. [M4] Proposition 2.1). □

In order to be able to relate the auxiliary function $\mathbf{z}$ with $\mathbf{x}$ we also need the following result:

Proposition 2. *The mapping $\mathbf{z} = (z^1, z^2)$ defined in (12) fulfils the relations*

$$c^{-1}|\nabla\mathbf{x}| \le |\mathbf{z}| \le c|\nabla\mathbf{x}| \quad \textit{on } B^+ \tag{17}$$

with some constant $c > 0$.

Proof. The right-hand inequality in (17) is obvious by definition. In order to prove the left-hand inequality we write (12) as

$$\mathbf{z} = \mathbf{A}(\mathbf{x}) \cdot \begin{pmatrix} x_w^1 \\ x_w^3 \end{pmatrix} + \mathbf{b}(\mathbf{x})x_w^2 \quad \text{on } B^+ \tag{18}$$

with

$$\mathbf{A} := \begin{pmatrix} -i\psi_{p^1} & i \\ 1 - iq\dot{\gamma} & \psi_{p^1} + \psi_{p^2}\dot{\gamma} \end{pmatrix}, \quad \mathbf{b} := \begin{pmatrix} -i\psi_{p^2} \\ \dot{\gamma} + iq \end{pmatrix}. \tag{19}$$

Pick $0 < \varepsilon < 1 - q_0$ arbitrarily. According to the normalization (6) we may choose $r = r(\varepsilon) > 0$ sufficiently small to ensure

$$|\det \mathbf{A}(\mathbf{p})| \ge 1 - \varepsilon > 0 \quad \text{for } \mathbf{p} \in \overline{\mathcal{B}_r}. \tag{20}$$

In particular, the inverse $\mathbf{A}^{-1}(\mathbf{p})$ exists on $\overline{\mathcal{B}_r}$, and we conclude

$$\begin{pmatrix} x_w^1 \\ x_w^3 \end{pmatrix} = \mathbf{A}^{-1}(\mathbf{x}) \cdot \mathbf{z} - \mathbf{A}^{-1}(\mathbf{x}) \cdot \mathbf{b}(\mathbf{x})x_w^2 \quad \text{on } B^+. \tag{21}$$

Computing

$$\mathbf{A}^{-1} \cdot \mathbf{b} = \frac{1}{\det \mathbf{A}} \begin{pmatrix} q - i[\psi_{p^1}\psi_{p^2} + (1 + \psi_{p^2}^2)\dot{\gamma}] \\ q(\psi_{p^1} + \psi_{p^2}\dot{\gamma}) + i(\psi_{p^2} - \psi_{p^1}\dot{\gamma}) \end{pmatrix},$$

the smallness (11) of q, inequality (20), and the normalization (6) imply

$$|\mathbf{A}^{-1}(\mathbf{p}) \cdot \mathbf{b}(\mathbf{p})| \le q_0 + \varepsilon \quad \text{for } \mathbf{p} \in \overline{\mathcal{B}_r}$$

with sufficiently small $r = r(\varepsilon) > 0$. Finally, we write the conformality relations in (1) as $\langle \mathbf{x}_w, \mathbf{x}_w \rangle = 0$ in B^+, which yields

$$|x_w^2|^2 \le |x_w^1|^2 + |x_w^3|^2 \quad \text{on } B^+.$$

With these estimates we conclude

$$\sqrt{|x_w^1|^2 + |x_w^3|^2} \le c|\mathbf{z}| + (q_0 + \varepsilon)\sqrt{|x_w^1|^2 + |x_w^3|^2} \quad \text{on } B^+$$

from (21), where $c > 0$ denotes a constant. Choosing e.g. $\varepsilon = \frac{1-q_0}{2}$, we hence obtain the claimed estimate (13) with an aligned $c > 0$. □

Combining Propositions 1 and 2, we arrive at the following

Lemma 1. *Let* $\mathbf{z} = (z^1, z^2)$ *be defined by (12). Set* $B := B_1(0)$, $B^- := B \setminus (B^+ \cup I)$ *and consider the reflected function*

$$\hat{\mathbf{z}}(w) := \begin{cases} \mathbf{z}(w), & w \in B^+ \\ \overline{\mathbf{z}(\overline{w})}, & w \in B^- \end{cases} \in C^1(B \setminus I, \mathbb{C}^2) \cap L_2(B, \mathbb{C}^2). \tag{22}$$

Then there exists $\mathbf{h} \in L_\infty(B, \mathbb{C}^2)$ *such that* $\hat{\mathbf{z}}$ *solves the equation*

$$\int_B \big(\langle \hat{\mathbf{z}}, \boldsymbol{\varphi}_{\overline{w}}\rangle + |\hat{\mathbf{z}}|^2 \langle \mathbf{h}, \boldsymbol{\varphi}\rangle\big)\, du\, dv = 0 \quad \textit{for all } \boldsymbol{\varphi} \in C^0_c(B, \mathbb{C}^2) \cap H^1_2(B, \mathbb{C}^2). \tag{23}$$

Proof. The assertion follows from the estimate

$$|\hat{\mathbf{z}}_{\overline{w}}| \le c|\hat{\mathbf{z}}|^2 \quad \text{on } B \setminus I, \tag{24}$$

which we will prove below. Indeed, defining

$$\mathbf{h}(w) := \begin{cases} |\hat{\mathbf{z}}(w)|^{-2}\hat{\mathbf{z}}_{\overline{w}}, & \text{for } w \in B \setminus I \text{ with } |\hat{\mathbf{z}}(w)| \neq 0 \\ 0, & \text{otherwise} \end{cases} \in L_\infty(B, \mathbb{C}^2),$$

we infer $\hat{\mathbf{z}}_{\overline{w}}(w) = |\hat{\mathbf{z}}(w)|^2 \mathbf{h}(w)$ away from isolated points in $B \setminus I$, because points $w \in B^+$ with $|\mathbf{z}(w)| = 0$ are exactly the isolated branch points of $\mathbf{x}$. If we multiply this relation with an arbitrary $\boldsymbol{\varphi} \in C^1_c(B, \mathbb{C}^2)$, integrate over $B^\pm_{(\varrho)} := \{w \in B^\pm \,:\, \pm v > \varrho\}$ and apply Gauss' integral theorem as well as the boundary condition, Proposition 1, we arive at (23) for such $\boldsymbol{\varphi}$. By a standard approximation argument we can also allow $\boldsymbol{\varphi} \in C^0_c(B, \mathbb{C}^2) \cap H^1_2(B, \mathbb{C}^2)$ in (23).

By showing (24), the proof will be completed. To this end, we reflect $\mathbf{x}$ trivially across I,

$$\hat{\mathbf{x}}(w) := \begin{cases} \mathbf{x}(w), & w \in B^+ \cup I \\ \mathbf{x}(\overline{w}), & w \in B^- \end{cases}. \tag{25}$$

Defining $\mathbf{A}, \mathbf{b} \in C^1(\overline{\mathcal{B}_r})$ by (19) and having (18) in mind, we now may write $\hat{\mathbf{z}}$ as

$$\hat{\mathbf{z}} = \mathbf{A}(\hat{\mathbf{x}}) \cdot \begin{pmatrix} \hat{x}^1_w \\ \hat{x}^3_w \end{pmatrix} + \mathbf{b}(\hat{\mathbf{x}})\, \hat{x}^2_w \quad \text{on } B^+ \tag{26}$$

and as

$$\hat{\mathbf{z}} = \overline{\mathbf{A}(\hat{\mathbf{x}})} \cdot \begin{pmatrix} \hat{x}^1_w \\ \hat{x}^3_w \end{pmatrix} + \overline{\mathbf{b}(\hat{\mathbf{x}})}\, \hat{x}^2_w \quad \text{on } B^-. \tag{27}$$

On the other hand, Rellich's system in (1) can be written as

$$\hat{\mathbf{x}}_{w\overline{w}} = \pm i \mathcal{H}(\hat{\mathbf{x}}) \hat{\mathbf{x}}_{\overline{w}} \wedge \hat{\mathbf{x}}_w \quad \text{on } B^\pm. \tag{28}$$

Differentiating (26), (27) and applying (28), we obtain

$$|\hat{\mathbf{z}}_{\overline{w}}| \le c|\nabla \hat{\mathbf{x}}|^2 \quad \text{on } B \setminus I$$

with some constant $c > 0$. Hence, Proposition 2 yields the asserted relation (24). □

Now the crucial step in the proof of Theorem 1 is the following

Lemma 2. *For any $\mu \in (0,1)$, the mapping $\hat{\mathbf{z}}$ defined in Lemma 1 can be extended to a mapping of class $C^\mu(B,\mathbb{C}^2)$ with the property $Im\,\hat{\mathbf{z}} = \mathbf{0}$ on I.*

Proof. We attempt to recover the steps in Section 3 of [M4], which were used there to prove an analogue result, namely Lemma 3.4.

1. At first, we prove $\hat{\mathbf{x}} \in C^\beta(B,\mathbb{R}^3)$ for some $\beta \in (0,1)$. To this end, we consider the function

$$\chi := \begin{cases} \hat{x}^3 - \psi(\hat{x}^1,\hat{x}^2) & \text{on } B^+ \cup I \\ -\hat{x}^3 + \psi(\hat{x}^1,\hat{x}^2) & \text{on } B^- \end{cases} . \tag{29}$$

Note that $\chi \in C^0(B) \cap H_2^1(B)$ is satisfied according to the boundary condition (2). Choose any disc $B_\varrho(w_0) \subset\subset B$ and define $\mathbf{y} = (y^1,y^2) \in C^\infty(B_\varrho(w_0),\mathbb{R}^2) \cap C^0(\overline{B_\varrho(w_0)},\mathbb{R}^2)$ as harmonic vector with boundary values

$$y^1 = \hat{x}^1, \quad y^2 = \chi \quad \text{on } \partial B_\varrho(w_0).$$

Setting

$$\boldsymbol{\varphi} := \begin{pmatrix} -i(\chi - y^2) \\ \hat{x}^1 - y^1 \end{pmatrix} \quad \text{on } \overline{B_\varrho(w_0)}, \qquad \boldsymbol{\varphi} := \mathbf{0} \quad \text{on } B \setminus \overline{B_\varrho(w_0)},$$

we obtain an admissible test function $\boldsymbol{\varphi} \in C_c^0(B,\mathbb{C}^2) \cap H_2^1(B,\mathbb{C}^2)$ for (23). We now insert $\boldsymbol{\varphi}$ and the relations (26), (27) for $\hat{\mathbf{z}}$ into (23) and use the special form (19) of $\mathbf{A}$ and $\mathbf{b}$. Writing $\boldsymbol{\xi} := (\hat{x}^1,\hat{x}^3)$, we then find

$$(1-d(r)) \int\limits_{B_\varrho(w_0)} |\boldsymbol{\xi}_w|^2\,du\,dv \;\le\; (q_0 + d(r)) \int\limits_{B_\varrho(w_0)} |\boldsymbol{\xi}_w|\,|\hat{x}^2_w|\,du\,dv$$
$$+c \int\limits_{B_\varrho(w_0)} |\mathbf{y}_w|\,|\hat{\mathbf{x}}_w|\,du\,dv + \int\limits_{B_\varrho(w_0)} |\hat{\mathbf{z}}|^2 |\mathbf{h}|\,|\boldsymbol{\varphi}|\,du\,dv$$

where $c > 0$ is a constant and $d(r)$, $0 < r \ll 1$, denotes some (possibly varying) positive function satisfying $d(r) \to 0\,(r \to 0+)$. By our global assumption (9), the maximum principle, and the normalization $\psi(0,0) = 0$ we further get $|\boldsymbol{\varphi}| \le d(r)$. Using the conformality relations as well as Proposition 2 we hence conclude

$$(1 - q_0 - d(r)) \int\limits_{B_\varrho(w_0)} |\hat{\mathbf{x}}_w|^2\,du\,dv \le c \int\limits_{B_\varrho(w_0)} |\mathbf{y}_w|\,|\hat{\mathbf{x}}_w|\,du\,dv.$$

Applying the inequality of Cauchy-Schwarz and assuming $d(r) \le \frac{1}{2}(1-q_0)$, we finally arrive at

$$\int\limits_{B_\varrho(w_0)} |\nabla\hat{\mathbf{x}}|^2\,du\,dv \le c \int\limits_{B_\varrho(w_0)} |\nabla\mathbf{y}|^2\,du\,dv \quad \text{for all discs } B_\varrho(w_0) \subset\subset B. \tag{30}$$

Note that there is a constant $c > 0$ with

$$c^{-1}|\nabla\hat{\mathbf{x}}| \le |\nabla(\hat{x}^1,\chi)| \le c|\nabla\hat{\mathbf{x}}| \quad \text{on } B$$

due to the conformality relations and the condition $\nabla\psi(0,0)=0$. Employing C. B. Morrey's Dirichlet growth theorem, we hence infer $\hat{\mathbf{x}}\in C^\beta(B,\mathbb{R}^3)$ for some $\beta\in(0,1)$ from (30).

2. Next we show: For any $\alpha\in[0,2\beta)$ and any compact subset $K\subset B$ we have
$$\int\limits_B |w-w_0|^{-\alpha}|\hat{\mathbf{z}}(w)|^2\,du\,dv\le c\quad\text{for all } w_0\in K, \tag{31}$$
where $c>0$ denotes a constant depending on α and K.
We fix some $w_0\in K$ and define χ as in (29). We consider
$$\boldsymbol{\psi}(w):=\begin{pmatrix}-i(\chi(w)-\chi(w_0))\\ \hat{x}^1(w)-\hat{x}^1(w_0)\end{pmatrix},\quad w\in B.$$
According to part 1 of the proof we have $\chi,\hat{x}^1\in C^\beta(B)$ and conclude
$$|\boldsymbol{\psi}(w)|\le c|w-w_0|^\beta,\quad w\in K. \tag{32}$$
Moreover, we can estimate (remember $\boldsymbol{\xi}=(\hat{x}^1,\hat{x}^3)$)
$$\begin{aligned}\langle\hat{\mathbf{z}},\boldsymbol{\psi}_{\overline{w}}\rangle &\ge |\boldsymbol{\xi}_w|^2-d(r)|\hat{\mathbf{x}}_w|^2-(q_0+d(r))|\boldsymbol{\xi}_w||\hat{x}^2_w|\\ &\ge (1-q_0-d(r))|\boldsymbol{\xi}_w|^2\ \ge\ c(1-q_0-d(r))|\hat{\mathbf{z}}|^2\quad\text{in } B,\end{aligned} \tag{33}$$
where we retained the notation of part 1 and used Proposition 2.
Now we choose some $\delta\in(0,\delta_0)$, $\delta_0:=\frac{1}{2}\mathrm{dist}(K,\partial B)$, and set
$$\gamma(w):=\begin{cases}\delta^{-\alpha}-\delta_0^{-\alpha}, & 0\le|w-w_0|<\delta\\ |w-w_0|^{-\alpha}-\delta_0^{-\alpha}, & \delta\le|w-w_0|<\delta_0\\ 0, & \delta_0\le|w-w_0|\end{cases}.$$
Then $\boldsymbol{\phi}:=\gamma\boldsymbol{\psi}\in C^0_c(B,\mathbb{C}^2)\cap H^1_2(B,\mathbb{C}^2)$ is admissible in (23) and relations (32), (33) as well as $|\langle\mathbf{h},\boldsymbol{\psi}\rangle|\le d(r)$ yield
$$c(1-q_0-d(r))\int\limits_B\gamma|\hat{\mathbf{z}}|^2\,du\,dv\le c\int\limits_{\delta<|w-w_0|<\delta_0}|w-w_0|^{-\alpha-1+\beta}|\hat{\mathbf{z}}|\,du\,dv. \tag{34}$$
We assume $d(r)\le\frac{1}{2}(1-q_0)$ and apply the inequalities
$$\int\limits_B\gamma|\hat{\mathbf{z}}|^2\,du\,dv\ge\int\limits_{\delta<|w-w_0|<\delta_0}|w-w_0|^{-\alpha}|\hat{\mathbf{z}}|^2\,du\,dv-\delta_0^{-\alpha}\int\limits_B|\hat{\mathbf{z}}|^2\,du\,dv$$
and
$$\begin{aligned}\int\limits_{\delta<|w-w_0|<\delta_0}|w-w_0|^{-\alpha-1+\beta}|\hat{\mathbf{z}}|\,du\,dv\ &\le\ \frac{\varepsilon}{2}\int\limits_{\delta<|w-w_0|<\delta_0}|w-w_0|^{-\alpha}|\hat{\mathbf{z}}|^2\,du\,dv\\ &\quad+\frac{1}{2\varepsilon}\int\limits_{\delta<|w-w_0|<\delta_0}|w-w_0|^{-\alpha-2+2\beta}\,du\,dv\end{aligned}$$

with sufficiently small $\varepsilon > 0$ to (34). Having $\int_B |\hat{\mathbf{z}}|^2\,du\,dv < +\infty$ as well as $2\beta > \alpha$ in mind, we arrive at

$$\int\limits_{\delta<|w-w_0|<\delta_0} |w-w_0|^{-\alpha}|\hat{\mathbf{z}}|^2\,du\,dv \le c$$

with some constant $c > 0$ which is independent of $w_0 \in K$ and $\delta \in (0, \delta_0)$. For $\delta \to 0+$ we obtain the asserted estimate (31).

3. Finally, it turns out that (31) is valid for $\alpha = 1$. This can be proved exactly as in [M4] Proposition 3.3 via an induction argument using the representation formula of Pompeiu and Vekua, namely

$$\hat{\mathbf{z}}(w) = \mathbf{y}(w) - \frac{1}{\pi}\int\limits_B \frac{|\hat{\mathbf{z}}(\zeta)|^2\mathbf{h}(\zeta)}{\zeta - w}\,d\xi\,d\eta, \quad w \in B; \quad \zeta = \xi + i\eta, \tag{35}$$

with some holomorphic vector $\mathbf{y} : B \to \mathbb{C}^2$. Hence $\hat{\mathbf{z}}$ is locally bounded in B. By applying E. Schmidt's inequality (see e.g. [DHT] pp. 219–221) to a local version of (35), we conclude $\hat{\mathbf{z}} \in C^\mu(B, \mathbb{C}^2)$ for any $\mu \in (0, 1)$, as asserted. The property $\mathrm{Im}(\hat{\mathbf{z}}) = \mathbf{0}$ on I is now an immediate consequence of Proposition 1. □

As the last preliminaries towards the proof of Theorem 1 we need two further lemmata; the first one is due to E. Heinz, S. Hildebrandt, and J.C.C. Nitsche and we present it in a special appropriate form:

Lemma 3. (Heinz–Hildebrandt–Nitsche)

(a) Let $f \in C^0(B^+, \mathbb{C})$ be given such that its square f^2 has a continuous extension to $B^+ \cup I$. Then f can be extended to a continuous function $f \in C^0(B^+ \cup I, \mathbb{C})$.

(b) Let $f \in C^0([-\varrho_0, \varrho_0], \mathbb{C})$ be given with some $\varrho_0 \in (0, 1)$. Suppose that $Re(f) \cdot Im(f) = 0$ on $[-\varrho_0, \varrho_0]$ is satisfied and that there exist numbers $c > 0$, $\alpha \in (0, 1]$ with

$$|f^2(u_1) - f^2(u_2)| \le c|u_1 - u_2|^{2\alpha} \quad \textit{for all } u_1, u_2 \in [-\varrho_0, \varrho_0]. \tag{36}$$

Then we have $f \in C^\alpha([-\varrho_0, \varrho_0], \mathbb{C})$.

Proof. We refer to the Lemmata 3 and 4 in [DHT] Section 2.7. □

The second of the announcend lemmata contains a regularity result for generalized analytic functions; we give its proof for the sake of completeness:

Lemma 4. *Let $z \in C^1(B^+, \mathbb{C}) \cap C^0(B^+ \cup I, \mathbb{C})$ be a solution of*

$$z_{\overline{w}} = g \quad \textit{in } B^+, \qquad Im\, z = h \quad \textit{on } [-\varrho_0, \varrho_0] \tag{37}$$

for some $\varrho_0 \in (0, 1)$. Then there hold:

(a) If $g \in C^0(B^+ \cup I, \mathbb{C})$ and $h \in C^\alpha([-\varrho_0, \varrho_0])$ for some $\alpha \in (0, 1)$, then we have $z \in C^\alpha(\overline{B^+_\varrho(0)}, \mathbb{C})$ for any $\varrho \in (0, \varrho_0)$.

(b) If $g \in C^\alpha(B^+ \cup I, \mathbb{C})$ and $h \in C^{1,\alpha}([-\varrho, \varrho])$ for some $\alpha \in (0,1)$, then we have $z \in C^{1,\alpha}(\overline{B_\varrho^+(0)}, \mathbb{C})$ for any $\varrho \in (0, \varrho_0)$.

Proof. 1. We first prove assertion (a). Fix some $\varrho \in (0, \varrho_0)$ and choose a test function $\phi \in C_c^\infty(B)$ with $\phi = 1$ in $\overline{B_\varrho(0)}$ and $\phi = 0$ in $B \setminus B_{\frac{\varrho+\varrho_0}{2}}(0)$ as well as a simply connected domain $B^+_{\frac{\varrho+\varrho_0}{2}}(0) \subset G \subset B^+_{\varrho_0}(0)$ with C^2-boundary. Let $\sigma : B \to G$ be a conformal mapping. Then the function $\tilde{z} := (\phi z) \circ \sigma \in C^1(B, \mathbb{C}) \cap C^0(\overline{B}, \mathbb{C})$ solves a boundary value problem

$$\tilde{z}_{\overline{w}} = \tilde{g} \quad \text{on } B, \qquad \operatorname{Im} \tilde{z} = \tilde{h} \quad \text{on } \partial B, \tag{38}$$

where $\tilde{g} \in C^0(\overline{B}, \mathbb{C})$, $\tilde{h} \in C^\alpha(\partial B)$ ist satisfied; here one has to use the well-known Kellogg-Warschawski theorem on the boundary behaviour of conformal mappings, see e.g. [P]. By subtracting a holomorphic function in B with boundary values $\tilde{h}$ we may assume $\tilde{h} \equiv 0$; note that this holomorphic function belongs to $C^\alpha(\overline{B}, \mathbb{C})$ by a well-known result of I. I. Privalov. Now, any solution of (38) with $\tilde{h} \equiv 0$ has the form

$$\tilde{z}(w) = -\frac{1}{\pi} \int\limits_B \frac{\tilde{g}(\zeta)}{\zeta - w}\, d\xi\, d\eta - \frac{w}{\pi} \overline{\int\limits_B \frac{\tilde{g}(\zeta)}{1 - \overline{w}\zeta}\, d\xi\, d\eta} + z_0, \quad w \in \overline{B}, \tag{39}$$

with some constant $z_0 \in \mathbb{R}$; see Theorem 2 in [S] Chap. IX, § 4. Defining the *Vekua-Operator*

$$T[\tilde{g}](w) := -\frac{1}{\pi} \int\limits_B \frac{\tilde{g}(\zeta)}{\zeta - w}\, d\xi\, d\eta, \quad w \in \mathbb{C},$$

we may rewrite (39) as

$$\tilde{z}(w) = T[\tilde{g}](w) + \overline{T[\tilde{g}]\Big(\frac{1}{\overline{w}}\Big)} + z_0, \quad w \in \overline{B}.$$

Well-known estimates for the Vekua-operator (see [V] Chap. I, § 6) now show $\tilde{z} \in C^\alpha(\overline{B})$ and hence $z \in C^\alpha(\overline{B_\varrho^+(0)}, \mathbb{C})$. This proves (a).

2. For the proof of claim (b) we repeat the construction above and note that, by (a), the right hand sides in (38) satisfy $\tilde{g} \in C^\alpha(\overline{B}, \mathbb{C})$, $\tilde{h} \in C^{1,\alpha}(\partial B)$. Subtracting a holomorphic function with boundary values $\tilde{h}$, which belongs to $C^{1,\alpha}(\overline{B}, \mathbb{C})$ by Privalov's theorem, we may again assume $\tilde{h} \equiv 0$. According to Theorem 2 in [S] Chap. IX, § 4 (see also [V] Chap. I, § 8) the solution (39) of this problem belongs to $C^{1,\alpha}(\overline{B}, \mathbb{C})$ and we conclude $z \in C^{1,\alpha}(\overline{B_\varrho^+(0)}, \mathbb{C})$, as asserted.

□

We are now prepared to give the proof of our main result, Theorem 1. To this end, we define a further auxiliary function, namely

$$z^3 := -(\dot{\gamma} + iq)x_w^1 + (1 - iq\dot{\gamma})x_w^2 + (\psi_{p^2} - \psi_{p^1}\dot{\gamma})x_w^3 \in C^1(B^+, \mathbb{C}) \cap H_2^1(B^+, \mathbb{C}) \tag{40}$$

with $q = q(\mathbf{x})$, $\dot{\gamma} = \dot{\gamma}(x^1)$, $\psi_{p^j} = \psi_{p^j}(x^1, x^2)$; remember the definitions of ψ, γ, and q in (7), (8), and (10). If we set $\boldsymbol{\zeta} := (\mathbf{z}, z^3) = (z^1, z^2, z^3) : B^+ \to \mathbb{C}^3$, we have the identity

$$\boldsymbol{\zeta}(w) = \mathbf{B}(\mathbf{x}(w)) \cdot \mathbf{x}_w(w), \quad w \in B^+, \tag{41}$$

where we abbreviated

$$\mathbf{B} := \begin{pmatrix} -i\psi_{p^1} & -i\psi_{p^2} & i \\ 1-iq\dot\gamma & \dot\gamma+iq & \psi_{p^1}+\psi_{p^2}\dot\gamma \\ -(\dot\gamma+iq) & 1-iq\dot\gamma & \psi_{p^2}-\psi_{p^1}\dot\gamma \end{pmatrix} \in C^1(\overline{\mathcal{B}_r},\mathbb{C}^{3\times3}). \tag{42}$$

Note that

$$\det\mathbf{B} = i(1+\dot\gamma^2)(1-q^2+|\nabla\psi|^2) \neq 0 \quad \text{on } \overline{\mathcal{B}_r}$$

is true according to the smallness condition (11). Hence, the inverse $\mathbf{B}^{-1}(\mathbf{p})$ exists for any $\mathbf{p}\in\overline{\mathcal{B}_r}$ and we have $\mathbf{B}^{-1}\in C^1(\overline{\mathcal{B}_r},\mathbb{C}^{3\times3})$.

We intend to employ the conformality relations, which now can be written as

$$0 = \langle \mathbf{x}_w,\mathbf{x}_w\rangle = \big\langle \mathbf{B}^{-1}(\mathbf{x})\boldsymbol\zeta, \mathbf{B}^{-1}(\mathbf{x})\boldsymbol\zeta\big\rangle = \langle\boldsymbol\zeta,\mathbf{C}(\mathbf{x})\boldsymbol\zeta\rangle \quad \text{on } B^+ \tag{43}$$

with the matrix $\mathbf{C} = (c_{ij})_{i,j=1,2,3} := \mathbf{B}^{-T}\cdot\mathbf{B}^{-1} \in C^1(\overline{\mathcal{B}_r},\mathbb{C}^{3\times3})$. A lengthy but straightforward computation yields

$$\begin{aligned} c_{11} &= -\frac{1-q^2}{1-q^2+|\nabla\psi|^2}, \\ c_{12} &= \frac{q(\psi_{p^2}-\psi_{p^1}\dot\gamma)}{(1+\dot\gamma^2)(1-q^2+|\nabla\psi|^2)} = c_{21}, \\ c_{13} &= -\frac{q(\psi_{p^1}+\psi_{p^2}\dot\gamma)}{(1+\dot\gamma^2)(1-q^2+|\nabla\psi|^2)} = c_{31}, \\ c_{22} &= \frac{1+\dot\gamma^2+(\psi_{p^2}-\psi_{p^1}\dot\gamma)^2}{(1+\dot\gamma^2)^2(1-q^2+|\nabla\psi|^2)}, \\ c_{23} &= -\frac{(\psi_{p^1}+\psi_{p^2}\dot\gamma)(\psi_{p^2}-\psi_{p^1}\dot\gamma)}{(1+\dot\gamma^2)^2(1-q^2+|\nabla\psi|^2)} = c_{32}, \\ c_{33} &= \frac{1+\dot\gamma^2+(\psi_{p^1}+\psi_{p^2}\dot\gamma)^2}{(1+\dot\gamma^2)^2(1-q^2+|\nabla\psi|^2)}. \end{aligned} \tag{44}$$

In particular, we have $\mathbf{C}:\overline{\mathcal{B}_r}\to\mathbb{R}^{3\times3}$. We are now ready to give the

Proof of Theorem 1. 1. We write (43) in the form

$$0 = \sum_{j,k=1}^{3} c_{jk}z^jz^k = c_{33}(z^3)^2 + 2(c_{13}z^1+c_{23}z^2)z^3 + \sum_{j,k=1}^{2} c_{jk}z^jz^k \quad \text{on } B^+,$$

where we abbreviated $c_{jk} = c_{jk}\circ\mathbf{x}$. Since $c_{33}>0$ holds on $\overline{\mathcal{B}_r}$ due to (44), we may rewrite this identity as

$$\Big(z^3 + \sum_{j=1}^{2}\frac{c_{j3}}{c_{33}}z^j\Big)^2 = \Big(\sum_{j=1}^{2}\frac{c_{j3}}{c_{33}}z^j\Big)^2 - \sum_{j,k=1}^{2}\frac{c_{jk}}{c_{33}}z^jz^k \quad \text{on } B^+. \tag{45}$$

By Lemma 2, we may extend the right hand side of (45) to a continuous function on $B^+\cup I$. Lemma 3 (a) thus yields that also $z^3+\sum_{j=1}^{2}\frac{c_{j3}}{c_{33}}z^j$ and, again due to Lemma 2, $\boldsymbol\zeta=(z^1,z^2,z^3)$ can be extended continuously to $B^+\cup I$. The definition (41) of $\boldsymbol\zeta$ as well as $\det\mathbf{B}\neq0$ now imply $\mathbf{x}\in C^1(B^+\cup I,\mathbb{R}^3)$.

2. Now we prove part (i) of the theorem. For fixed $\varrho_0 \in (0,1)$ and any $\mu \in (0,1)$ the right hand side of (45) belongs to $C^\mu([-\varrho_0,\varrho_0],\mathbb{C})$ according to Lemma 2 and $\mathbf{x} \in C^1(B^+ \cup I, \mathbb{R}^3)$. In addition, the imaginary part of the right hand side vanishes on $[-\varrho_0,\varrho_0]$ due to $\mathrm{Im}(z^1) = \mathrm{Im}(z^2) = 0$ on I (see again Lemma 2) and to $\mathbf{C} : \overline{\mathcal{B}_r} \to \mathbb{R}^{3\times 3}$ as shown above. Hence, the function $f = z^3 + \sum_{j=1}^2 \frac{c_{j3}}{c_{33}} z^j \in C^0(I,\mathbb{C})$ satisfies the assumptions of Lemma 3 (b) for any $\alpha \in (0,\frac{1}{2})$. We conclude $f \in C^\alpha([-\varrho_0,\varrho_0],\mathbb{C})$ and by Lemma 2 also $\boldsymbol{\zeta} \in C^\alpha([-\varrho_0,\varrho_0],\mathbb{C}^3)$ for any $\alpha \in (0,\frac{1}{2})$. If we differentiate (41) w.r.t. $\overline{w}$ and apply Rellich's system (28) we obtain

$$\boldsymbol{\zeta}_{\overline{w}} = \mathbf{g} \quad \text{on } B^+ \qquad \text{with some} \quad \mathbf{g} \in C^0(B^+ \cup I, \mathbb{C}^3).$$

Consequently, we may apply Lemma 4 (a) to $\boldsymbol{\zeta}$ and find $\boldsymbol{\zeta} \in C^\alpha(\overline{B_\varrho^+(0)},\mathbb{C}^3)$ as well as $\mathbf{x} \in C^{1,\alpha}(\overline{B_\varrho^+(0)},\mathbb{R}^3)$ for any $\varrho \in (0,\varrho_0)$ and any $\alpha \in (0,\frac{1}{2})$. Since we localized around an arbitrary point $w_0 \in I$, the proof of Theorem 1 (a) is completed.

3. For the proof of Theorem 1 (ii) we assume $\mathcal{S} \in C^{2,\beta}$, $\mathbf{Q} \in C^{1,\beta}(\mathbb{R}^3,\mathbb{R}^3)$ with some $\beta \in (0,1)$. Then we also have $\mathbf{B} \in C^{1,\beta}(\overline{\mathcal{B}_r},\mathbb{R}^{3\times 3})$ and by part (i) we know $\mathbf{x} \in C^{1,\frac{1}{4}}(B^+,\mathbb{R}^3)$. Set $\gamma := \min\{\frac{1}{4},\beta\}$, define $\mathbf{z} = (z^1,z^2)$ by (12) and differentiate these equations w.r.t. $\overline{w}$. Then we obtain

$$\mathbf{z}_{\overline{w}} = \mathbf{g}_0 \quad \text{on } B^+, \qquad \mathrm{Im}\,\mathbf{z} = \mathbf{0} \quad \text{on } I$$

with some $\mathbf{g}_0 \in C^\gamma(B^+ \cup I, \mathbb{C}^2)$. From Lemma 4 (b) we thus conclude $\mathbf{z} \in C^{1,\gamma}([-\varrho,\varrho],\mathbb{C}^2)$ for any $\varrho \in (0,1)$. In particular, the right hand side of equation (45) belongs to $C^1([-\varrho,\varrho],\mathbb{C})$ and Lemma 3 (b) shows $\boldsymbol{\zeta} \in C^{\frac{1}{2}}([-\varrho,\varrho],\mathbb{C}^3)$ for any $\varrho \in (0,1)$. Now Lemma 4 (a) can be applied to get $\boldsymbol{\zeta} \in C^{\frac{1}{2}}(\overline{B_\varrho^+(0)},\mathbb{C}^3)$ and we finally arrive at $\mathbf{x} \in C^{1,\frac{1}{2}}(B^+ \cup I,\mathbb{R}^3)$, as asserted. □

We conclude the paper with the

Proof of Theorem 2. We choose a branch point $w_0 \in I$ and assume $\mathbf{x}(w_0) \in \partial S$; compare Remark 4 above. We localize as above – note especially $w_0 \mapsto 0$ – and define $\mathbf{z} = (z^1,z^2)$ by (12). Reflecting $\mathbf{z}$ as in (22), the resulting function $\hat{\mathbf{z}} : B \to \mathbb{C}^2$ satisfies $\hat{\mathbf{z}} \in C^1(B \setminus I,\mathbb{C}^2) \cap C^0(B,\mathbb{C}^2)$ and $\mathrm{Im}\,\hat{\mathbf{z}} = \mathbf{0}$ on I according to Lemma 2.

Now choose an arbitrary domain $D \subset\subset B$ with piecewise smooth boundary. Then the arguments leading to formula (23) in Lemma 1 yield

$$\frac{1}{2i}\oint_{\partial D} \langle \hat{\mathbf{z}}, \boldsymbol{\varphi}\rangle\, dw = \int_D \big(\langle \hat{\mathbf{z}}, \boldsymbol{\varphi}_{\overline{w}}\rangle + |\hat{\mathbf{z}}|^2 \langle \mathbf{h}, \boldsymbol{\varphi}\rangle\big)\, du\, dv \quad \text{for all } \boldsymbol{\varphi} \in C^1(B,\mathbb{C}^2);$$

here $\mathbf{h} : B \to \mathbb{C}^2$ denotes some bounded function. According to the boundedness of $\hat{\mathbf{z}}$ on D we find a constant $c > 0$ such that

$$\Big|\oint_{\partial D} \langle \hat{\mathbf{z}}, \boldsymbol{\varphi}\rangle\, dw\Big| \le 2\int_D \big(|\boldsymbol{\varphi}_{\overline{w}}| + c|\boldsymbol{\varphi}|\big)|\hat{\mathbf{z}}|\, du\, dv \quad \text{for all } \boldsymbol{\varphi} \in C^1(B,\mathbb{C}^2).$$

The Hartman-Wintner technique – see e.g. Theorem 1 in [DHT] Section 3.1 – now implies the existence of some $m \in \mathbb{N}$ and some vector $\hat{\mathbf{b}} \in \mathbb{C}^2 \setminus \{\mathbf{0}\}$ such that

$$\hat{\mathbf{z}}(w) = \hat{\mathbf{b}} w^m + o(|w|^m) \quad \text{as } w \to 0. \tag{46}$$

Note here that $\hat{\mathbf{z}}$ cannot vanish identically in B since, otherwise, we would have $\nabla \mathbf{x} \equiv \mathbf{0}$ near w_0 due to Proposition 2; this is impossible by our assumption $\mathbf{x} \not\equiv$ const as can be easily seen by employing the well known asymptotic expansions at interior branch points.

Next we define z^3 by (40) and consider $\boldsymbol{\zeta} = (z^1, z^2, z^3) = (\mathbf{z}, z^3)$, which can be extended to a continuous function on $\overline{B_\varrho^+(0)}$ for any $\varrho \in (0,1)$, according to part 2 in the proof of Theorem 1. In addition, we recall the relation (45), where the quantities $c_{jk} = c_{jk} \circ \mathbf{x}$ are continuous functions on $\overline{B^+}$.

Now we multiply (45) by w^{-2m} and let $w \in \overline{B_\varrho^+(0)}$ tend to 0. Due to (46), the right hand side and hence also the left hand side converges. Applying (46) again as well as a variant of Lemma 3 (i), we find $w^{-m} z^3(w) \to b^3$ as $w \to 0$ with some limit $b^3 \in \mathbb{C}$. Setting $\mathbf{b} := (\hat{\mathbf{b}}, b^3) \in \mathbb{C}^3$, we conclude

$$\boldsymbol{\zeta}(w) = \mathbf{b} w^m + o(|w|^m) \quad \text{as } w \to 0. \tag{47}$$

This relation finally yields the announced expansion (5) according to $\mathbf{x}_w = (\mathbf{B}^{-1} \circ \mathbf{x})\boldsymbol{\zeta}$; see (41) and recall $\det \mathbf{B} \neq 0$. The relation $\langle \mathbf{a}, \mathbf{a} \rangle = 0$ is now a direct consequence of the conformality relations and (5). $\square$

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