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BERNSTEIN RESULTS FOR SYMMETRIC MINIMAL SURFACES OF CONTROLLED GROWTH

by

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BERNSTEIN RESULTS FOR SYMMETRIC MINIMAL SURFACES OF CONTROLLED GROWTH

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Abstract. We prove that there is no entire solution of the symmetric minimal surface equation which is of sublinear growth. This result is extended to parametric and non-parametric minimizers of the corresponding variational integral.

0. Introduction

By a well known result of Bernstein [BS] every entire classical solution u of the minimal surface equation

$$\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right)=0$$

in $\mathbb{R}^2$, has to be an affine-linear function. In fact this theorem was shown to hold up to dimension 7 by Fleming [FW], De Giorgi [DG], Almgren [AF] and J. Simons [SJ], while there exist nonlinear entire solutions in $\mathbb{R}^n, n \geq 8$, as was first discovered by Bombieri - De Giorgi - Giusti [BDG]. Many more non-affine examples were constructed by L. Simon [SL2].

On the other hand Moser [MJ] proved that every entire solution u of the minimal surface equation in $\mathbb{R}^n$, n arbitrary, is affine linear, provided $|Du|_{0,\mathbb{R}^n}$ is finite, and it follows from the a-priori gradient estimate of Bombieri - De Giorgi - Miranda [BDGM] that this is already the case if u grows at most linearly, in the sense that

$$u(x) \leq C(1+|x|) \text{ for some } C>0 \text{ and all } x \in \mathbb{R}^n.$$

Ecker and Huisken [EH] extended Moser's result by requiring instead of boundedness only sublinear growth of the gradient Du, that is

$$|Du(x)| = \mathrm{o}(|x|) \quad \text{as} \quad |x| \to \infty.$$

Optimal results of this type were proved by L. Simon [SL2], [SL3].

In this paper we consider entire solutions of the *symmetric minimal surface equation* (in short: s.m.s.e.)

$$(*) \qquad \operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right)=\frac{\alpha}{u\sqrt{1+|Du|^2}}$$

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where $\alpha > 0$ denotes some positive number. $(*)$ is the Euler-equation of the variational integral

$$E(u) = \int u^\alpha \sqrt{1 + |Du|^2}\, dx,$$

which, for $\alpha = m \in \mathbb{N}$ and positive $u : \Omega \to \mathbb{R}^+$, describes, up to a constant factor, the area of the rotated graph

$$\mathcal{M}_{\text{rot}} = \left\{(x, u(x)\omega) \in \mathbb{R}^n \times \mathbb{R}^{m+1}; x \in \Omega \subset \mathbb{R}^n \text{ and } \omega \in S^m\right\}$$

where $S^m \subset \mathbb{R}^{m+1}$ denotes the unit m-sphere, see e. g. the computation in [DU7].

A different interpretation for $(*)$ with $\alpha = 1$ in the two-dimensional case was already given by Poisson [PS], who considered $(*)$ as a model equation for an ideal *"heavy surface of constant mass density"* which is exposed to a vertical gravitational field. Furthermore, architects consider $(*)$ as a model equation for a so called *"hanging roof"*, which is of importance for the constructions of *"perfect domes"* or *"cupolas"*, see the discussion in [OF] and the literature cited therein.

The *symmetric* (or *"singular"*) *minimal surface equation* $(*)$ is an equation of mean curvature type, with mean curvature H given by

$$H(u, Du) = \frac{\alpha}{u\sqrt{1 + |Du|^2}}\,,$$

whence H is a-priori not bounded, nor can a solution u of $(*)$ be of class C^2 in a neighbourhood of a point x_0 with $u(x_0) = 0$. Thus we typically consider either classical positive solutions, or weak Lipschitz solutions $u \geq 0$ of the s.m.s.e. For the existence of classical solutions of $(*)$ with prescribed boundary values we refer to the papers by Dierkes - Huisken [DH] and Dierkes [DU6].

On the other hand, it is easily checked that the cones

$$c_n^\alpha(x) := \sqrt{\frac{\alpha}{n-1}}\,(x_1^2 + \ldots + x_n^2)^{\frac{1}{2}} = \sqrt{\frac{\alpha}{n-1}}\,|x|$$

are classical solutions of $(*)$ on $\mathbb{R}^n - \{0\}$ and weak Lipschitz-solutions on all of $\mathbb{R}^n$, for every $\alpha > 0$, $n \geq 2$. For a complete classification of these cones concerning their minimizing properties and for the construction of nonaffine entire C^∞-solution asymptotic to these cones, we refer to the papers by Dierkes [DU1], [DU2], [DU3].

In view of these remarks the following result is optimal.

Theorem 1. *There is no entire nonnegative solution $u \in C^{0,1}(\mathbb{R}^n)$ of the symmetric minimal surface equation $(*)$ satisfying*

$$u(x) = o(|x|) \quad \text{as} \quad |x| \to \infty.$$

(Here $\alpha > 0, n \geq 2$ are arbitrary).

We also prove a version of Theorem 1 for less regular, local minimizers of the integral E in $\mathbb{R}^n$.

Theorem 2. *Let $\alpha > 0$ and $u \in BV^{1+\alpha}_{+,\text{loc}}(\mathbb{R}^n)$ be a local minimizer of E in $\mathbb{R}^n$ which is of sublinear growth. Then $u \equiv 0$.*

Here the class $BV_+^{1+\alpha}(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is open and $\alpha > 0$ is defined by

$$BV_+^{1+\alpha}(\Omega) := \left\{ u \in L_{1+\alpha}(\Omega); u \geq 0 \text{ and } u^{1+\alpha} \in BV(\Omega) \right\}.$$

It is the natural function space on which the integral

$$E(u) = \int_\Omega u^\alpha \sqrt{1 + |Du|^2}\, dx$$

can be defined (as a measure) and also minimized, cp. the papers by Bemelmans and Dierkes [BD] and [DU3]. Note that $\frac{1}{2}$-Hölder-continuity is the optimal regularity for minimizers of $E(\cdot)$ that can be expected in general, see the examples by Dierkes [DU1], [DU2]. Recently T. Tennstädt [TT1][TT2] proved $\frac{1}{2}$-Hölder-continuity for every minimizer in dimensions $n \leq 6$. Again, by the examples constructed in [DU1], [DU2] it follows that Theorem 2 is optimal of its type.

Thirdly we prove an analogous result for Caccioppoli sets minimizing the parametric energy functional

$$\mathcal{E}(U) = \int |x_{n+1}|^\alpha\, |D\varphi_U|\,,$$

see chapters 2 and 3 for the pertinent definitions.

Theorem 3. *Let $\alpha > 0$ and $U \subset \mathbb{R}^{n+1}$ be a Caccioppoli set which locally minimizes the integral $\mathcal{E}(\cdot)$ in $\mathbb{R}^{n+1}$ and which is of sublinear growth. Then U is the half-space $\{(x, x_{n+1}) \in \mathbb{R}^n \times \mathbb{R}\ ;\ x_{n+1} \leq 0\}$.*

Finally we consider certain types of *"exterior"* solutions of the s. m. s. e. $(*)$ which possibly vanish on a set of positive measure.

Theorem 4. *Let $\alpha > 1$ and $n \geq 2$ be arbitrary. There is no non-trivial non-negative function $u \in H^1_{1,\mathrm{loc}}(\mathbb{R}^n) \cap C^0(\mathbb{R}^n)$ which solves the symmetric minimal surface equation $(*)$ weakly in $\mathbb{R}^n - \{u = 0\}$, where the coincidence set $\{u = 0\}$ is supposed to be bounded and which is of sublinear growth in the sense that*

$$u(x) = o(|x|) \quad as \quad |x| \to \infty.$$

The examples constructed in [DU1], [DU2] are of class $H^1_{p,\mathrm{loc}}(\mathbb{R}^n) \cap C^{0,\frac{1}{2}}(\mathbb{R}^n)$, $\forall p < 2$, vanish on balls $\mathcal{B}_R(0) \subset \mathbb{R}^n$ and are of linear growth at infinity. Hence Theorem 4 is optimal.

Further Bernstein type results for stable solutions of $(*)$ in small dimensions were proved in [DU5].

The proofs of Theorems 1, 2, 3 and 4 follow from suitable monotonicity and area estimates given in Sections 2 and 3. The Theorems are proved in Section 4.

1. Preliminaries

We here consider quite generally integer multiplicity n-rectifiable varifolds $v = v(M, \Theta)$ in $\mathbb{R}^{n+1}$ (in the sense of Allard and Simon [SL1]), briefly *"integer n-varifolds"*, that is – modulo n-dimensional Hausdorff-measure zero – a countably

n-rectifiable $\mathcal{H}^n$-measurable subset M of $\mathbb{R}^{n+1}$ together with an integer valued positive and locally integrable function Θ on M. Associated to the varifold v is the Radon measure $\mu_v := \mathcal{H}^n \llcorner \Theta$ i.e. $\mu_v(A) = \int_A \Theta \, d\mathcal{H}^n = \int_{A\cap M} \Theta \, d\mathcal{H}^n$ for any $\mathcal{H}^n$ measurable set $A \subset \mathbb{R}^{n+1}$, where we have put $\Theta \equiv 0$ outside of M. In particular we have in mind varifolds (with multiplicity $\Theta = 1$) given by the *reduced boundary* $\partial^* E$ of a Caccioppoli set $E \subset \mathbb{R}^{n+1}$. Recall that $E \subset U \subset \mathbb{R}^{n+1}$, U open, is a set of locally *finite perimeter* (or *"Caccioppoli set"*) in U, if E is $\mathcal{L}^{n+1}$-measurable and if the characteristic function φ_E of E has locally finite bounded variation in U, $\varphi_E \in BV_{\mathrm{loc}}(U)$. If $E \subset \mathbb{R}^{n+1}$ has locally finite perimeter in $U \subset \mathbb{R}^{n+1}$ there is a Radon measure $\mu_E = |D\varphi_E|$ on U and a $|D\varphi_E|$ measurable function $\nu = (\nu_1, \ldots, \nu_{n+1})$ (the generalized inward unit normal) with $\|\nu(x)\| = 1$ for $|D\varphi_E|$ a. e. $x \in U$ and such that for every $g = (g_1, \ldots, g_{n+1}) \in C^1_c(U, \mathbb{R}^{n+1})$ we have

$$\begin{aligned}\int_{E\cap U} \operatorname{div} g \, d\mathcal{L}^{n+1} &= -\int_U (g\cdot\nu)|D\varphi_E| \\ &= -\int_U g \cdot D\varphi_E\end{aligned}$$

$D\varphi_E$ denoting the vector measure $\nu|D\varphi_E|$. Furthermore the reduced boundary $\partial^* E$ of a Caccioppoli set E is given by

$$\partial^* E = \left\{ x \in U; \quad \lim_{\rho\to 0} \frac{\int_{B_\rho(x)} \nu |D\varphi_E|}{\int_{B_\rho(x)} |D\varphi_E|} \text{ exists and has length equal to } 1 \right\}.$$

In particular we have $|D\varphi_E| = |D\varphi_E| \llcorner \partial^* E = \mathcal{H}^n \llcorner \partial^* E$, $\partial^* E$ is countably n-rectifiable and each point $x \in \partial^* E$ has an approximate tangent space T_x with multiplicity 1 given by

$$T_x = \left\{ y \in \mathbb{R}^{n+1};\ y \cdot \nu_E(x) = 0 \right\}, \quad \text{where} \quad \nu_E(x) := \lim_{\rho\to 0} \frac{\int_{B_\rho(x)} \nu|D\varphi_E|}{\int_{B_\rho(x)} |D\varphi_E|},$$

see [GE] and [SL1] for more discussion and proofs.

Now let $v = v(M, \Theta)$ be a rectifiable n-varifold in $U \subset \mathbb{R}^{n+1}, U$ open, and consider the functional

$$\mathcal{E}_\alpha(M) = \int_M |x_{n+1}|^\alpha \, d\mu_v \quad , \alpha > 0.$$

The first variation can be computed e.g. as in Simon [SL1] or [DU4]; for convenience we sketch the proof.

To this end consider a one parameter family $\Phi_t, -1 \le t \le 1$, of diffeomorphisms of $U \subset \mathbb{R}^{n+1}$ with the following properties,

i) $\Phi_t(x) = \Phi(t,x) \in C^2\left((-1,1) \times U, U\right)$
ii) $\Phi_0 \equiv Id_{|U}$
iii) $\Phi_t(x) = x$ for all $t \in [-1,1]$ and every $x \in U - K$ for some compact set $K \subset U$.

Put $X(x) := \dfrac{\partial \Phi}{\partial t}(t,x)_{|t=0} \in C^1_c(U, \mathbb{R}^{n+1})$ to denote the initial velocity vector for $\Phi(t,x)$ and let $\Phi_{t\#}v$ denote the image varifold $\Phi_{t\#}v = v\left(\Phi_t(M), \Theta \circ \Phi_t^{-1}\right)$. The general area-formula ([SL1]) yields

$$\mathcal{E}_\alpha\left(\Phi_{t\#}(v \llcorner K)\right) = \int_{M\cap K} |\Psi_t^{n+1}|^\alpha J\Psi_t \cdot \Theta d\mathcal{H}^n$$

where we have put $\Psi_t := \Phi_{t|M\cap K}$, K compact, $K \subset U$ and $J\Psi_t$ denotes the Jacobian of Ψ_t. By definition the first variation is given by

$$\delta\mathcal{E}_\alpha(v, X) := \frac{d}{dt}\,\mathcal{E}_\alpha\left(\Phi_{t\#}(v \llcorner K)\right)_{|t=0}.$$

Proposition 1. *Let $v = v(M, \Theta)$ be an integer n-rectifiable varifold, $\Phi_t(x) = \Phi(t, x)$ and $X(x) = \frac{\partial}{\partial t}\Phi(t, x)|_{t=0}$ be as above. Suppose either $M \subset \mathbb{R}^n \times \mathbb{R}^+, \mathbb{R}^+ := \{t > 0\}$, or $\alpha > 1$, then the first variation of $\mathcal{E}_\alpha$ is given by*

$$\delta\mathcal{E}_\alpha(v) = \int_{M\cap K} |x_{n+1}|^\alpha \left(\operatorname{div}_M X(x) + \alpha\,\frac{X^{n+1}(x)}{x_{n+1}}\right) d\mu_v$$

where X^{n+1} denotes the $(n+1)$-st component of the vector field $X = (X^1, \ldots, X^{n+1})$.

Proof. For convenience we sketch the argument and refer to [SL1] [DU4] and [DHT] chapter 3.2 for more detailed calculations. By standard arguments one finds for the Jacobian $J\Psi_t$ the development

$$J\Psi_t = 1 + t\operatorname{div}_M X + \mathcal{O}(t^2), \text{ while}$$
$$|\Psi_t^{n+1}(x)|^\alpha = |x_{n+1}|^\alpha \left\{1 + \alpha\, t\frac{X^{n+1}(x)}{x_{n+1}} + \mathcal{O}(t^2)\right\}.$$

The first variation formula now follows by computing the coefficient of t in the product $|\Psi_t^{n+1}(x)|^\alpha \cdot J\Psi_t$. □

Definition 1. *The varifold $v = v(M, \Theta)$ is called stationary in $U \subset \mathbb{R}^{n+1}$, U open, if*

$$\int_M |x_{n+1}|^\alpha \left(\operatorname{div}_M X(x) + \alpha\,\frac{X^{n+1}(x)}{x_{n+1}}\right) d\mu = 0 \tag{1}$$

holds for all vector fields $X(x) = \big(X^1(x), \ldots, X^{n+1}(x)\big) \in C_c^1(U, \mathbb{R}^{n+1})$.

Remark. Here we either assume $\alpha > 1$ or $M \subset \mathbb{R}^n \times \mathbb{R}^+$ (or $M \subset \mathbb{R}^n \times \mathbb{R}^-$, $\mathbb{R}^- = \{t < 0\}$).

Proposition 2. *Let $M \subset \mathbb{R}^{n+1}$ be a C^2-hypersurface and $U \subset \mathbb{R}^{n+1}$ be an open set, such that $M \cap U \neq \emptyset$, $\partial M \cap U = \emptyset$ and $\mathcal{H}^n(M \cap K) < \infty$ for each compact set $K \subset U$. Then M is stationary in U if and only if the mean curvature $H = H(x)$, $x \in M \cap U$, with respect to the unit normal $\nu = (\nu_1, \ldots, \nu_{n+1}) = \nu(x)$ satisfies the Euler equation*

$$|x_{n+1}|^\alpha H(x) = \alpha |x_{n+1}|^\alpha \frac{\nu_{n+1}}{x_{n+1}}. \tag{2}$$

Remarks.

i) Clearly, if $M \subset \mathbb{R}^n \times \mathbb{R}^+$, (2) is equivalent to $H(x) = \alpha\,\frac{\nu_{n+1}}{x_{n+1}}$, $\forall x \in M$, and also, if $M = \operatorname{graph}(u)$ for some positive function $u : \Omega \to \mathbb{R}^+$, to the symmetric minimal surface equation

$$\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right)=\frac{\alpha}{u\sqrt{1+|Du|^2}}. \tag{3}$$

On the other hand, given a stationary C^2 hypersurface $M\subset\mathbb{R}^n\times\mathbb{R}$ and a point $y_0:=(\hat{y}_0,0)\in M, \hat{y}_0\in\mathbb{R}^n$ with the property that every ball $B_\varepsilon(y_0)\subset\mathbb{R}^{n+1}$, $\varepsilon>0$, contains points $y_\varepsilon\in M\cap B_\varepsilon(y_0)$ with $(y_\varepsilon)_{n+1}\neq 0$ then we can conclude

$$\lim_{\varepsilon\to 0}\left(\frac{\alpha\,\nu_{n+1}(y_\varepsilon)}{y_\varepsilon^{n+1}}\right)=H(y_0)\text{ exists;}$$

in particular $\nu_{n+1}(y_0)=0$. Hence M intersects the coordinate plane $\{x_{n+1}=0\}$ vertically an y_0 and can be written locally at y_0 as a graph $x_1=f(x_2,\dots,x_{n+1})$ say (which satisfies some singular elliptic p.d.e.).

ii) The coordinate plane $\{x_{n+1}=0\}$ satisfies (2) (with $\alpha>1$) but is not a solution of (3).

iii) There are Lipschitz hypersurface solutions of (2) given by the union of any vertical half-plane and the corresponding half-plane of the coordinate plane $\{x_{n+1}=0\}$.

iv) There exist (Lipschitz-)continuous piecewise C^2-hypersurfaces which are $\mathcal{H}^n$-a. e. solutions of (2) (for $\alpha>1$), namely the union of an n-ball $\mathcal{B}_R(0)\subset\mathbb{R}^n\times\{0\}$ and a C^2-hypersurface in $\mathbb{R}^n\times\mathbb{R}^+$ with boundary $\partial\mathcal{B}_R(0)$ given by the graph of a particular $\frac{1}{2}$-Hölder continuous function $u:\mathbb{R}^n-\mathcal{B}_R(0)\to\mathbb{R}^+\cup\{0\}$. See the work of Dierkes [DU1].

Proof of Proposition 2. Suppose $M\subset\mathbb{R}^{n+1}$ is stationary in U and let $X(x):=\xi(x)\cdot\nu(x)$, where $\xi\in C^1_c(U,\mathbb{R})$ is arbitrary and ν is some unit normal on M. Then $\operatorname{div}_M X=\xi\operatorname{div}_M\nu=-\xi H$ and hence (2) follows from (1) and a standard device. On the other hand, if $M\in C^2$ satisfies (2) and $X\in C^1_c(U,\mathbb{R}^{n+1})$ is given arbitrarily, we decompose $X=X^\perp+X^\top$ into its normal part $X^\perp=(X\cdot\nu)\,\nu$ and the tangential part $X^\top\in T_xM$ respectively and compute $\operatorname{div}_M X^\perp=(X\cdot\nu)\operatorname{div}_M\nu=-H(X\cdot\nu)$. Therefore we have

$$|x_{n+1}|^\alpha\operatorname{div}_M X^\perp=-|x_{n+1}|^\alpha H(X\cdot\nu)=-\alpha|x_{n+1}|^\alpha\frac{\nu_{n+1}}{x_{n+1}}(X\cdot\nu) \tag{4}$$

by (2). Furthermore we find

$$\begin{aligned}|x_{n+1}|^\alpha\operatorname{div}_M X^\top&=\operatorname{div}_M(|x_{n+1}|^\alpha X^\top)-\nabla_M(|x_{n+1}|^\alpha)\,X^\top\\&=\operatorname{div}_M\{|x_{n+1}|^\alpha X^\top\}-\alpha\frac{|x_{n+1}|^\alpha}{x_{n+1}}(\nabla_M x_{n+1}\cdot X^\top)\\&=\operatorname{div}_M\{|x_{n+1}|^\alpha X^\top\}-\alpha\frac{|x_{n+1}|^\alpha}{x_{n+1}}X^{n+1}\\&\qquad+\alpha\frac{|x_{n+1}|^\alpha}{x_{n+1}}\nu_{n+1}\,(X\cdot\nu)\end{aligned} \tag{5}$$

where we have used the relation

$$\begin{aligned}\nabla_M x_{n+1} \cdot X^\top &= (\mathrm{e}_{n+1} - (\mathrm{e}_{n+1} \cdot \nu)\nu) \cdot X^\top \\ &= (\mathrm{e}_{n+1} - (\mathrm{e}_{n+1} \cdot \nu)\nu) \cdot X \\ &= X^{n+1} - \nu_{n+1}(X \cdot \nu),\end{aligned}$$

denoting by e_{n+1} the vector $(0, \ldots, 0, 1) \in \mathbb{R}^{n+1}$. Concluding we finally obtain from (4) and (5) the identity

$$\begin{aligned}&|x_{n+1}|^\alpha \left(\mathrm{div}_M\, X + \alpha \frac{X^{n+1}(x)}{x_{n+1}} \right) \\ &= \mathrm{div}_M \{ |x_{n+1}|^\alpha X^\top \} - \alpha \frac{|x_{n+1}|^\alpha}{x_{n+1}} X^{n+1} + \alpha \frac{|x_{n+1}|^\alpha}{x_{n+1}} \nu_{n+1}(X \cdot \nu) \\ &\quad - \alpha \frac{|x_{n+1}|^\alpha}{x_{n+1}} \nu_{n+1}(X \cdot \nu) + \alpha \frac{|x_{n+1}|^\alpha X^{n+1}}{x_{n+1}} \\ &= \mathrm{div}_M \{ |x_{n+1}|^\alpha X^\top \}.\end{aligned}$$

Hence (1) follows from the divergence theorem since $X^\top$ has compact support on M. □

Proposition 3. *Let $u \in C^{0,1}(\mathbb{R}^n)$ be a weak nonnegative solution of the symmetric minimal surface equation $(*)$ in $\mathbb{R}^n$ with $\alpha > 0$. Then $M = \mathrm{graph}(u) \subset \mathbb{R}^{n+1}$ is stationary in $\mathbb{R}^{n+1}$, i.e.*

$$\int_M x_{n+1}^\alpha \left\{ \mathrm{div}_M\, X(x) + \alpha \frac{X^{n+1}(x)}{x_{n+1}} \right\} d\mathcal{H}^n(x) = 0$$

holds for all vectorfields $X = (X^1, \ldots, X^{n+1}) \in C^1_c(\mathbb{R}^{n+1}, \mathbb{R}^{n+1})$.

Remark. Note that here it is not assumed $\alpha > 1$ although the level set $\{u = 0\}$ might be nonempty. In fact we show existence of the integral in this case, even if $\alpha \in (0, 1]$.

Proof. Since $M = \{(x, u(x)) \in \mathbb{R}^n \times \mathbb{R}\}$ is the Lipschitz image of $\mathbb{R}^n$ it is countably n-rectifiable and by Schauder theory we have $u \in C^\infty(\{u > 0\})$. Whence the mean curvature of $M \cap \mathbb{R}^n \times \{t > 0\}$ is simply

$$H(x) = \alpha \frac{\nu_{n+1}}{x_{n+1}} = \frac{\alpha}{u\sqrt{1 + |Du|^2}}, \; x = (x_1, \ldots, x_{n+1})$$

and by Proposition 2 it follows that M is stationary in $\mathbb{R}^n \times \{t > 0\}$ that is we have the relation

$$\int_M x_{n+1}^\alpha \left\{ \mathrm{div}_M\, X + \alpha \frac{X^{n+1}}{x_{n+1}} \right\} d\mathcal{H}^n(x) = 0 \tag{6}$$

for all vectorfields $X \in C^1_c(\mathbb{R}^n \times \{t > 0\}, \mathbb{R}^{n+1})$ (and, clearly, for all $X \in C^1_c(\mathbb{R}^n \times \{t \neq 0\}, \mathbb{R}^{n+1})$ since $u \geq 0$).

By assumption $u \in C^{0,1}(\mathbb{R}^n) = H^1_{\infty,\mathrm{loc}}(\mathbb{R}^n)$ is a solution of the equation

$$\int_{\mathbb{R}^n} \left\{ \frac{Du\, D\varphi}{\sqrt{1 + |Du|^2}} + \frac{\alpha\varphi}{u\sqrt{1 + |Du|^2}} \right\} dx = 0$$

for all $\varphi \in C^1_c(\mathbb{R}^n)$, and $|Du| \in L_{\infty,\mathrm{loc}}(\mathbb{R}^n)$ together with a standard test function argument imply that

$$\frac{1}{u} \in L_{1,\mathrm{loc}}(\mathbb{R}^n), \text{ whence also } \mathcal{L}^n(\{u=0\}) = \mathcal{H}^n(\{u=0\}) = 0.$$

For $\varepsilon > 0$ consider a smooth cutoff function $\eta_\varepsilon : \mathbb{R} \to \mathbb{R}$ given by the conditions $\eta_\varepsilon(t) = 1$, for $|t| \geq 3\varepsilon$, $\eta_\varepsilon(t) = 0$, for $|t| \leq \varepsilon$ and $0 \leq \eta_\varepsilon \leq 1$, $|\eta'_\varepsilon(t)| \leq \frac{1}{\varepsilon}$ for all t, hence $\eta_\varepsilon \to 1$ a.e. as $\varepsilon \to 0$. Furthermore let $X \in C^1_c(\mathbb{R}^{n+1}, \mathbb{R}^{n+1})$ be an arbitrary vectorfield and suppose $\operatorname{supp} X \subset B_R(0) \subset \mathbb{R}^{n+1}$. The truncated vectorfield $X_\varepsilon(x) := \eta_\varepsilon(x_{n+1}) \cdot X(x)$ is admissible in (6) and since

$$\operatorname{div}_M X_\varepsilon(x) = \eta_\varepsilon(x_{n+1}) \operatorname{div}_M X + X(x) \cdot \eta'_\varepsilon(x_{n+1}) \cdot \nabla_M x_{n+1}$$

we get the relation

$$\int_{M\cap B_R} x^\alpha_{n+1} \Big\{ \eta_\varepsilon(x_{n+1}) \operatorname{div}_M X + X(x)\eta'_\varepsilon(x_{n+1}) \nabla_M x_{n+1} + \alpha \frac{X^{n+1}(x)}{x_{n+1}} \eta_\varepsilon(x_{n+1}) \Big\} \, d\mathcal{H}^n(x) = 0$$

for every $\varepsilon > 0$. The second integral can be estimated as follows

$$\begin{aligned} &\Big| \int_{M\cap B_R} x^\alpha_{n+1} \eta'_\varepsilon(x_{n+1}) X(x) \cdot \nabla_M x_{n+1} \, d\mathcal{H}^n(x) \Big| \\ &\leq \sup_{M\cap B_R} |X| \int_{M\cap B_R \cap \{\varepsilon \leq x_{n+1} \leq 3\varepsilon\}} x^\alpha_{n+1} \cdot \frac{1}{\varepsilon} \, d\mathcal{H}^n(x) \\ &\leq 3 \sup_{M\cap B_R} |X| \int_{M\cap B_R \cap \{\varepsilon \leq x_{n+1} \leq 3\varepsilon\}} x^{\alpha-1}_{n+1} \, d\mathcal{H}^n(x) \\ &\leq 3\|X\|_{0,B_R} \int_{\mathcal{B}_R(0)\cap\{0\leq u \leq 3\varepsilon\}} u^{\alpha-1} \sqrt{1+|Du|^2} \, dx \\ &\leq 3\|X\|_{0,B_R} \left\{1 + \|Du\|^2_{0,\mathcal{B}_R}\right\}^{\frac{1}{2}} \|u^{-1}\|_{1,\mathcal{B}_R} \cdot (3\varepsilon)^\alpha \\ &\to 0, \text{ as } \varepsilon \to 0, \end{aligned}$$

since $u^{-1} \in L_{1,\mathrm{loc}}(\mathbb{R}^n)$.

Observe in particular that the function $x^{\alpha-1}_{n+1}$ is integrable w.r.t. n-dimensional Hausdorff-measure over $M \cap B_R$ for all $\alpha \geq 0$. In addition, since $\eta_\varepsilon(x_{n+1}) \to 1$ $\mathcal{H}^n$-a.e. on $M \cap B_R$ (recall $\mathcal{H}^n(\{u = 0\}) = 0$), we infer from Lebesgue's dominated convergence theorem

$$\int_{M\cap B_R} x^\alpha_{n+1} \eta_\varepsilon(x_{n+1}) \operatorname{div}_M X(x) \, d\mathcal{H}^n(x) \to \int_{M\cap B_R} x^\alpha_{n+1} \operatorname{div}_M X(x) \, d\mathcal{H}^n(x)$$

and

$$\int_{M\cap B_R} \alpha \, x^{\alpha-1}_{n+1} X^{n+1}(x) \eta_\varepsilon(x_{n+1}) \, d\mathcal{H}^n(x) \to \int_{M\cap B_R} x^{\alpha-1}_{n+1} X^{n+1}(x) \, d\mathcal{H}^n(x)$$

both as $\varepsilon \to 0$. In conclusion we have

$$\int_{M\cap B_R} x^\alpha_{n+1} \Big\{ \operatorname{div}_M X(x) + \alpha \frac{X^{n+1}(x)}{x_{n+1}} \Big\} \, d\mathcal{H}^n(x) = 0$$

for arbitrary $X \in C_c^1(\mathbb{R}^{n+1}, \mathbb{R}^{n+1})$ compactly supported in the ball $B_R(0) \subset \mathbb{R}^{n+1}$. $\square$

Similarly we prove for $\alpha > 1$

Proposition 3′. *Let* $\alpha > 1$ *and* $u : \mathbb{R}^n \to \mathbb{R}_0^+ = \{t \geq 0\}$, $u \in H^1_{1,\mathrm{loc}}(\mathbb{R}^n) \cap C^0(\mathbb{R}^n)$, *be a weak solution of the s.m.s.e. (∗) in* $\mathbb{R}^n - \{u = 0\}$. *Then* $M := \mathrm{graph}(u)$ *is stationary in* $\mathbb{R}^{n+1}$.

Remarks.

i) Here we have in mind exterior solutions of (3) in $(\mathbb{R}^n - \overline{\Omega})$, where $\Omega \subset \mathbb{R}^n$ is bounded and open, which in addition satisfy $u = 0$ on $\overline{\Omega}$. Recall that there are even minima u for E of this type, where $\Omega = \mathcal{B}_R(0)$ is a ball and $u \in C^\infty(\mathbb{R}^n - \overline{B_R(0)} \cap C^{0,\frac{1}{2}}(\mathbb{R}^n) \cap H^1_{p,\mathrm{loc}}(\mathbb{R}^n)$, $\forall p < 2$, cp. [DU2]. Recently, Tennstädt [TT1][TT2] proved that every local minimizer u of E is of class $H^1_{1,loc} \cap C^{0,\frac{1}{2}}$, if $n \leq 6$.
ii) It was recently shown by Tennstädt [TT1][TT3] that, for minimizing functions u, the zero set $\{u = 0\}$ has locally finite perimeter and is locally mean convex.

Proof. By assumption the set $\{u > 0\}$ is open and classical regularity theory implies $u \in C^2(\{u > 0\})$. Furthermore $u \in H^1_{1,\mathrm{loc}}(\mathbb{R}^n) \subset BV_{\mathrm{loc}}(\mathbb{R}^n)$, whence the subgraph $U := \{(x,t) \in \mathbb{R}^n \times \mathbb{R};\, t < u(x)\}$ has locally finite perimeter given by $\int \sqrt{1 + |Du|^2}\, dx$ and $M = \partial^* U = \mathrm{graph}(u)$ is n-rectifiable. Invoking Proposition 2 we obtain that $M = \mathrm{graph}(u)$ is stationary in $\mathbb{R}^n \times \{t \neq 0\} \subset \mathbb{R}^{n+1}$ and a similar argument as the one given in the proof of Proposition 3, using that now $\alpha > 1$ is assumed, finishes the proof. $\square$

2. Monotonicity formulae

We here give two versions of the monotonicity formula; namely one for stationary varifolds and – somewhat differently – another formula for minimizing boundaries.

First assume that $v = v(M, \Theta)$ is stationary in $U \subset \mathbb{R}^{n+1}$, i.e. we have the identity

$$\int_M |x_{n+1}|^\alpha \left(\mathrm{div}_M X(x) + \alpha \frac{X^{n+1}(x)}{x_{n+1}} \right) d\mathcal{H}^n(x) = 0$$

for all differentiable vectorfields $X = (X^1, \ldots, X^{n+1})$ with compact support in U. We choose the standard test function $X(x) := \gamma(r)(x - \xi)$, where $\xi \in U$ is fixed, $r := |x - \xi|$ and $\gamma \in C^1(\mathbb{R})$ with $\gamma'(t) \leq 0$, $\forall t \in \mathbb{R}$, $\gamma(t) = 1$ for $t \leq \frac{\rho}{2}$, $\gamma(t) = 0$ for $t \geq \rho$ and $\overline{B_\rho(\xi)} \subset U$. Standard calculations (cf. [SL1] and [DHT]) yield

$$\mathrm{div}_M X(x) = \mathrm{div}_M \left(\gamma(r)(x - \xi)\right) = \gamma(r)\, \mathrm{div}_M (x - \xi) + \gamma'(r) \nabla_M r \cdot (x - \xi) \tag{7}$$

and since

$$\nabla_M r = \nabla_M |x - \xi| = \frac{(x - \xi)^\top}{|x - \xi|}$$

we have

$$\nabla_M r(x-\xi) = r\frac{(x-\xi)^\top}{|x-\xi|}\ \frac{(x-\xi)^\top}{|x-\xi|} = r\left[1-\left(\frac{(x-\xi)^\perp}{|x-\xi|}\right)^2\right] = r[1-|Dr^\perp|^2],$$

where $Dr = \frac{(x-\xi)}{|x-\xi|}$ denotes the gradient of r.

Furthermore

$$\begin{aligned}
\operatorname{div}_M(x-\xi) &= \sum_{j=1}^{n+1} \mathrm{e}_j \cdot \nabla_M(x_j-\xi_j) = \sum_{j=1}^{n+1} \mathrm{e}_j\, \mathrm{e}_j^\top \\
&= \sum_{j=1}^{n+1} \mathrm{e}_j(\mathrm{e}_j - \mathrm{e}_j^\perp) = (n+1) - \sum_{j=1}^{n+1} (\mathrm{e}_j^\perp)^2 \\
&= (n+1) - \sum_{j=1}^{n+1} [(\nu\, \mathrm{e}_j)\cdot \nu]^2 = (n+1) - 1 \\
&= n
\end{aligned} \tag{8}$$

since $\mathrm{e}_j = \mathrm{e}_j^\top + \mathrm{e}_j^\perp$ and $\nu\, \mathrm{e}_j = \nu_j = \nu\, \mathrm{e}_j^\perp$, $\mathrm{e}_1, \ldots, \mathrm{e}_{n+1}$ denoting the standard basis of $\mathbb{R}^{n+1}$. By (7), (8) and the first variation formula we find

$$\operatorname{div}_M X = n\gamma(r) + \gamma'(r)\, r(1-|Dr^\perp|^2)$$

whence

$$n\int_M |x_{n+1}|^\alpha \gamma(r)\, d\mu_v + \int_M |x_{n+1}|^\alpha \gamma'(r)\, r(1-|Dr^\perp|^2)\, d\mu_v \\ + \alpha \int_M |x_{n+1}|^\alpha x_{n+1}^{-1} \gamma(r)(x_{n+1}-\xi_{n+1})\, d\mu_v = 0,$$

or

$$(n+\alpha)\int_M |x_{n+1}|^\alpha \gamma(r)\, d\mu_v + \int_M |x_{n+1}|^\alpha r\gamma'(r)\, d\mu_v \\ = \alpha\int_M |x_{n+1}|^\alpha x_{n+1}^{-1}\gamma(r)\xi_{n+1} d\mu_v + \int_M |x_{n+1}|^\alpha \gamma'(r)\, r|Dr^\perp|^2 d\mu_v. \tag{9}$$

Now we take $\gamma(r) := \Phi\left(\frac{r}{\rho}\right)$ with $\Phi \in C^1(\mathbb{R})$ satisfying $\Phi(t) = 1$ if $t \le \frac{1}{2}$, $\Phi(t) = 0$ if $t \ge 1$, as well as $0 \le \Phi(t) \le 1$ and $\Phi'(t) \le 0$ for all $t \in \mathbb{R}$. Then

$$r\gamma'(r) = r\Phi'\left(\frac{r}{\rho}\right)\frac{1}{\rho} = -\rho\frac{\partial}{\partial\rho}\Phi\left(\frac{r}{\rho}\right)$$

and (9) yields

$$(n+\alpha)\int_M |x_{n+1}|^\alpha \Phi\left(\frac{r}{\rho}\right) d\mu_v - \rho\int_M |x_{n+1}|^\alpha \frac{\partial}{\partial\rho}\Phi\left(\frac{r}{\rho}\right) d\mu_v = \\ \alpha\int_M |x_{n+1}|^\alpha x_{n+1}^{-1}\Phi\left(\frac{r}{\rho}\right)\xi_{n+1} d\mu_v - \rho\int_M |x_{n+1}|^\alpha \frac{\partial}{\partial\rho}\Phi\left(\frac{r}{\rho}\right)|Dr^\perp|^2 d\mu_v.$$

Putting

$$
\begin{aligned}
I(\rho) &:= \int_M |x_{n+1}|^\alpha \Phi\left(\frac{r}{\rho}\right) d\mu_v \\
L(\rho) &:= \int_M |x_{n+1}|^\alpha x_{n+1}^{-1}\xi_{n+1}\Phi\left(\frac{r}{\rho}\right) d\mu_v \quad \text{and} \\
J(\rho) &:= \int_M |x_{n+1}|^\alpha \Phi\left(\frac{r}{\rho}\right) |Dr^\perp|^2 \, d\mu_v
\end{aligned}
$$

we infer the equation

$$(n+\alpha)I(\rho) - \rho I'(\rho) = \alpha L(\rho) - \rho J'(\rho)$$

and since

$$
\begin{aligned}
\frac{d}{d\rho}\left[\rho^{-(n+\alpha)}I(\rho)\right] &= -(n+\alpha)\rho^{-(n+\alpha+1)}I(\rho) + \rho^{-(n+\alpha)}I'(\rho) \\
&= -\rho^{-(n+\alpha+1)}\left[(n+\alpha)I - \rho I'\right]
\end{aligned}
$$

this implies the differential equation

$$\frac{d}{d\rho}\left(\rho^{-(n+\alpha)}I(\rho)\right) = \rho^{-(n+\alpha)}J'(\rho) - \alpha\rho^{-(n+\alpha+1)}L(\rho).$$

Integration between $0 < \sigma < \rho$ yields

$$\rho^{-(n+\alpha)}I(\rho) - \sigma^{-(n+\alpha)}I(\sigma) = \int_\sigma^\rho \tau^{-n-\alpha}J'(\tau)\,d\tau - \alpha\int_\sigma^\rho \tau^{-n-\alpha-1}L(\tau)\,d\tau$$

and upon partial integration of the first integral, then letting Φ tend to the characteristic function of the interval $(-\infty, 1)$ and finally applying Fubini's theorem, we conclude the monotonicity formula

$$
\begin{aligned}
&\rho^{-(n+\alpha)}\int_{B_\rho(\xi)} |x_{n+1}|^\alpha d\mu_v - \sigma^{-(n+\alpha)}\int_{B_\sigma(\xi)} |x_{n+1}|^\alpha d\mu_v \\
&= \int_{B_\rho - B_\sigma(\xi)} |x_{n+1}|^\alpha \frac{|Dr^\perp|^2}{r^{n+\alpha}}\,d\mu_v - \frac{\alpha\xi_{n+1}}{n+\alpha}\int_{B_\rho}\frac{|x_{n+1}|^\alpha}{x_{n+1}}\left[\frac{1}{r_\sigma^{n+\alpha}} - \frac{1}{\rho^{n+\alpha}}\right] d\mu_v
\end{aligned}
\tag{10}
$$

where $r_\sigma := \max(r, \sigma)$.

In particular, if $\xi_{n+1} = 0$ we have the identity

$$
\begin{aligned}
\sigma^{-(n+\alpha)}\int_{B_\sigma(\xi)} |x_{n+1}|^\alpha d\mu_v &= \rho^{-(n+\alpha)}\int_{B_\rho(\xi)} |x_{n+1}|^\alpha d\mu_v \\
&\quad - \int_{B_\rho - B_\sigma} |x_{n+1}|^\alpha \frac{|Dr^\perp|^2}{r^{n+\alpha}}\,d\mu_v
\end{aligned}
\tag{11}
$$

and the inequality

$$\sigma^{-(n+\alpha)}\int_{B_\sigma(\xi)} |x_{n+1}|^\alpha d\mu_v \le \rho^{-(n+\alpha)}\int_{B_\rho(\xi)} |x_{n+1}|^\alpha d\mu_v, \tag{12}$$

holding true for all $0 < \sigma \le \rho$ with $\overline{B_\sigma(\xi)} \subset U$.

We have thus proved

Proposition 4. *Suppose $v = v(M,\Theta)$ is stationary in $U \subset \mathbb{R}^{n+1}$ and $B_\rho(\xi) \subset\subset U$. Then we have the monotonicity formula (10), and if $\xi = (\xi_1, \dots, \xi_n, 0)$ the formulae (11) or (12) holding true.*

Remark. In general we assume $\alpha > 1$ in the definition of stationarity; however if $M = \operatorname{graph} u$, where $u \geq 0$ is some Lipschitz-solution of the s.m.s.e. $(*)$ then, because of Proposition 3, $\alpha > 0$ is sufficient in this case. In particular we then also have the monotonicity formulae for all $\alpha > 0$ and $M =$ graph of a Lipschitz solution u. Similarly, if v is given by the reduced boundary of a minimizing set $E \subset \mathbb{R}^{n+1}$, then we conclude a monotonicity formula for all $\alpha > 0$ directly from the minimizing property of v, rather then differentiating the functional as in Proposition 4, see Proposition 6. To show this we consider n-rectifiable varifolds $v = v(M,\Theta)$ given by the reduced boundary $\partial^* E$ of a Caccioppoli set $E \subset \mathbb{R}^{n+1}$ which locally minimizes the functional

$$\mathcal{E}(U) = \int |x_{n+1}|^\alpha \, |D\varphi_U|, \quad \alpha > 0,$$

in $\mathbb{R}^{n+1}$, i.e. we have

$$\int_\Omega |x_{n+1}|^\alpha \, |D\varphi_E| \leq \int_\Omega |x_{n+1}|^\alpha \, |D\varphi_F|$$

for any bounded open set $\Omega \subset \mathbb{R}^{n+1}$ and all sets $F \subset \mathbb{R}^{n+1}$ with locally finite perimeter such that $F \Delta E \subset\subset \Omega$. In other words, if we introduce the quantities $\mathrm{N} = \mathrm{N}(E,\Omega)$ by

$$\mathrm{N}(E,\Omega) := \inf \left\{ \int_\Omega |x_{n+1}|^\alpha |D\varphi_F| ; F \text{ has finite perimeter in } \Omega \text{ and } F\Delta E \subset\subset \Omega \right\}$$

and the *"indicator"* function $\Psi = \Psi(E,\Omega)$ by

$$\Psi(E,\Omega) := \int_\Omega |x_{n+1}|^\alpha |D\varphi_E| - \mathrm{N}(E,\Omega),$$

we consider $E \subset \mathbb{R}^{n+1}$, so that

$$\Psi(E,\Omega) = 0 \text{ for all open sets } \Omega \subset \mathbb{R}^{n+1}.$$

The following result immediately implies the monotonicity formula for minimizing boundaries, see also Giusti [GE] Lemma 5.8 for a similar estimate.

Proposition 5. *Let $E \subset \mathbb{R}^{n+1}$ have finite perimeter in a ball $B_R(0) \subset \mathbb{R}^{n+1}$. Then we have for all balls $B_\sigma(0) \subset B_\rho(0) \subset\subset B_R(0)$ the estimate*

$$\left(\int_{B_\rho - B_\sigma} |x_{n+1}|^\alpha \frac{|x \cdot D\varphi_E|}{|x|^{n+\alpha+1}} \right)^2 \leq 2 \left(\int_{B_\rho - B_\sigma} |x_{n+1}|^\alpha \frac{|D\varphi_E|}{|x|^{n+\alpha}} \right) \cdot \left\{ (n+\alpha) \int_\sigma^\rho r^{-n-\alpha-1} \Psi(E, B_r)\, dr + \rho^{-n-\alpha} \int_{B_\rho} |x_{n+1}|^\alpha |D\varphi_E| - \sigma^{-n-\alpha} \int_{B_\sigma} |x_{n+1}|^\alpha |D\varphi_E| \right\}$$

where $\alpha > 0$ *and* $B_\sigma = B_\sigma(0), B_\rho = B_\rho(0)$.

Remark. The same result holds for arbitrary balls $B_\sigma \subset\subset B_\rho(\xi) \subset B_R(0)$ with center $\xi = (\xi_1, \dots, \xi_n, 0)$ lying on the coordinate hyperplane $\{x_{n+1} = 0\}$.

Proof of Proposition 5. Let ϕ_E^ε be a mollification of the characteristic function φ_E with the properties

$$\begin{aligned} &\int_{B_r} |\varphi_E - \phi_E^\varepsilon| \, d\mathcal{H}^n \to 0, \text{ as } \varepsilon \to 0, \text{ and} \\ &\int_{B_r} |x_{n+1}|^\alpha |D\phi_E^\varepsilon| \, dx \to \int_{B_r} |x_{n+1}|^\alpha |D\varphi_E|, \text{ as } \varepsilon \to 0 \end{aligned} \tag{13}$$

for almost all $r \in [0, R]$, (cp. [MF] Thm. 12.3).

Define

$$\varphi_{E_{B_r}}(x) := \begin{cases} \varphi_E\left(r\frac{x}{|x|}\right) & \text{, if } |x| \le r \\ \varphi_E(x) & \text{, if } |x| > r \end{cases}$$

and

$$\eta_r^\varepsilon(x) := \phi_E^\varepsilon\left(r\frac{x}{|x|}\right).$$

Observe first that

$$\begin{aligned} \int_{B_r} |\eta_r^\varepsilon - \varphi_{E_{B_r}}| \, dx &= \int_0^r \int_{\partial B_\rho} |\eta_r^\varepsilon - \varphi_{E_{B_r}}| \, d\mathcal{H}^n \, d\rho \\ &= \int_0^r \left(\frac{\rho}{r}\right)^n \int_{\partial B_r} |\eta_r^\varepsilon - \varphi_{E_{B_r}}| \, d\mathcal{H}^n d\rho \\ &= \frac{r}{n+1} \int_{\partial B_r} |\phi_E^\varepsilon - \varphi| \, d\mathcal{H}^n \\ &\to 0 \text{ as } \varepsilon \to 0 \text{ f.a.a. } r \in [0, R] \end{aligned} \tag{14}$$

whence by lower semicontinuity also

$$\begin{aligned} \int_{B_r} |x_{n+1}|^\alpha |D\varphi_E| - \Psi(E, B_r) &\le \int_{B_r} |x_{n+1}|^\alpha |D\varphi_{E_{B_r}}| \\ &\le \liminf_{\varepsilon \to 0} \int_{B_r} |x_{n+1}|^\alpha |D\eta_r^\varepsilon| \, dx. \end{aligned} \tag{15}$$

From the definition of η_r^ε we compute

$$D\eta_r^\varepsilon(x) = r\left(\frac{D\phi_E^\varepsilon\left(r\frac{x}{|x|}\right)}{|x|} - \frac{\left(D\phi_E^\varepsilon(r\frac{x}{|x|}) \cdot x\right)}{|x|^3} \cdot x\right)$$

and therefore

$$\int_{B_r} |x_{n+1}|^\alpha |D\eta_r^\varepsilon|\,dx$$

$$= r\int_{B_r} |x_{n+1}|^\alpha \left\{ |x|^{-2}\left|D\phi_E^\varepsilon\left(r\frac{x}{|x|}\right)\right|^2 - |x|^{-4}\left(x\cdot D\phi_E^\varepsilon\left(r\frac{x}{|x|}\right)\right)^2\right\}^{\frac12} dx$$

$$= r\int_0^r\int_{\partial B_\tau} |x_{n+1}|^\alpha |x|^{-1}\left|D\phi_E^\varepsilon\left(r\frac{x}{|x|}\right)\right|\cdot\left\{1-\frac{\left(x\cdot D\phi_E^\varepsilon(r\frac{x}{|x|})\right)^2}{|x|^2|D\phi_E^\varepsilon(r\frac{x}{|x|})|^2}\right\}^{\frac12} d\mathcal{H}^n d\tau.$$

Using the transformation $x = \dfrac{\tau}{r}y$ we find

$$\int_{B_r} |x_{n+1}|^\alpha |D\eta_r^\varepsilon|\,dx$$

$$= r\int_0^r\int_{\partial B_r} |y_{n+1}|^\alpha |y|^{-1}\left(\frac{\tau}{r}\right)^{\alpha-1}|D\phi_E^\varepsilon(y)|\left\{1-\frac{(y\cdot D\phi_E^\varepsilon(y))^2}{|y|^2|D\phi_E^\varepsilon(y)|^2}\right\}^{\frac12}\left(\frac{\tau}{r}\right)^n d\mathcal{H}^n\,d\tau$$

(16)
$$\le r\int_0^r\left(\frac{\tau}{r}\right)^{n+\alpha-1}\int_{\partial B_r} |x_{n+1}|^\alpha r^{-1}|D\phi_E^\varepsilon|\left\{1-\frac{(x\cdot D\phi_E^\varepsilon(x))^2}{|x|^2|D\phi_E^\varepsilon(x)|^2}\right\}^{\frac12} d\mathcal{H}^n\,d\tau$$

$$\le \frac{r}{n+\alpha}\int_{\partial B_r} |x_{n+1}|^\alpha |D\phi_E^\varepsilon(x)|\left\{1-\frac12\,\frac{(x\cdot D\phi_E^\varepsilon(x))^2}{|x|^2|D\phi_E^\varepsilon(x)|^2}\right\} d\mathcal{H}^n.$$

Now multiply (15) by $r^{-n-\alpha-1}$, integrate over r from σ to ρ and then employ (16)

$$\int_\sigma^\rho r^{-n-\alpha-1}\left(\int_{B_r}|x_{n+1}|^\alpha|D\varphi_E| - \Psi(E,B_r)\right)dr$$

$$\le \liminf_{\varepsilon\to0}\int_\sigma^\rho r^{-n-\alpha-1}\int_{B_r}|x_{n+1}|^\alpha|D\eta_r^\varepsilon|\,dx\,dr$$

$$\le \liminf_{\varepsilon\to0}\left\{\frac{1}{n+\alpha}\int_\sigma^\rho r^{-n-\alpha}\int_{\partial B_r}|x_{n+1}|^\alpha|D\phi_E^\varepsilon(x)|\,d\mathcal{H}^n\,dr\right.$$

$$\left. -\frac{1}{2(n+\alpha)}\int_\sigma^\rho r^{-n-\alpha}\int_{\partial B_r}|x_{n+1}|^\alpha\frac{(x\cdot D\phi_E^\varepsilon(x))^2}{|x|^2|D\phi_E^\varepsilon(x)|}\,d\mathcal{H}^n\,dr\right\}$$

$$= \frac{1}{n+\alpha}\liminf_{\varepsilon\to0}\left\{\rho^{-n-\alpha}\int_{B_\rho}|x_{n+1}|^\alpha|D\phi_E^\varepsilon(x)|\,dx - \sigma^{-n-\alpha}\int_{B_\sigma}|x_{n+1}|^\alpha|D\phi_E^\varepsilon(x)|\,dx\right.$$

$$+(n+\alpha)\int_\sigma^\rho r^{-n-\alpha-1}\int_{B_r}|x_{n+1}|^\alpha|D\phi_E^\varepsilon(x)|\,dx\,dr$$

$$\left.-\frac12\int_\sigma^\rho r^{-n-\alpha}\int_{\partial B_r}|x_{n+1}|^\alpha\frac{(x\cdot D\phi_E^\varepsilon(x))^2}{|x|^2|D\phi_E^\varepsilon(x)|}\,d\mathcal{H}^n\,dr\right\},$$

where in the last step we have used an integration by parts. Rearranging terms we get

$$\begin{aligned}
&\limsup_{\varepsilon\to 0}\frac{1}{2(n+\alpha)}\int_{B_\rho-B_\sigma}|x_{n+1}|^\alpha\frac{(x\cdot D\phi_E^\varepsilon(x))^2}{|x|^{n+\alpha+2}|D\phi_E^\varepsilon(x)|}\,dx\\
&\le-\int_\sigma^\rho r^{-n-\alpha-1}\int_{B_r}|x_{n+1}|^\alpha|D\varphi_E|\,dr+\int_\sigma^\rho r^{-n-\alpha-1}\Psi(B_r)\,dr\\
&+\frac{1}{(n+\alpha)}\liminf_{\varepsilon\to 0}\Bigg\{\rho^{-n-\alpha}\int_{B_\rho}|x_{n+1}|^\alpha|D\phi_E^\varepsilon(x)|\,dx\\
&\qquad-\sigma^{-n-\alpha}\int_{B_\sigma}|x_{n+1}|^\alpha|D\phi_E^\varepsilon(x)|\,dx\\
&\qquad+(n+\alpha)\int_\sigma^\rho r^{-n-\alpha-1}\int_{B_r}|x_{n+1}|^\alpha|D\phi_E^\varepsilon(x)|\,dx\,dr\Bigg\}.
\end{aligned}\tag{17}$$

On the other hand we apply Schwarz' inequality to obtain

$$\begin{aligned}
&\left(\int_{B_\rho-B_\sigma}|x_{n+1}|^\alpha\frac{|x\cdot D\phi_E^\varepsilon(x)|}{|x|^{n+\alpha+1}}\,dx\right)^2\\
&\le\left(\int_{B_\rho-B_\sigma}|x_{n+1}|^\alpha\frac{|D\phi_E^\varepsilon(x)|}{|x|^{n+\alpha}}\,dx\right)\left(\int_{B_\rho-B_\sigma}|x_{n+1}|^\alpha\frac{(x\cdot D\phi_E^\varepsilon(x))^2}{|x|^{n+\alpha+2}|D\phi_E^\varepsilon(x)|}\,dx\right)
\end{aligned}$$

and estimate the second factor with the help of (17). This yields the inequality

$$\begin{aligned}
&\limsup_{\varepsilon\to 0}\left(\int_{B_\rho-B_\sigma}|x_{n+1}|^\alpha\frac{|D\phi_E^\varepsilon(x)\cdot x|}{|x|^{n+\alpha+1}}\,dx\right)^2\\
&\le\limsup_{\varepsilon\to 0}2(n+\alpha)\int_{B_\rho-B_\sigma}|x_{n+1}|^\alpha\frac{|D\phi_E^\varepsilon(x)|}{|x|^{n+\alpha}}\,dx\Bigg\{-\int_\sigma^\rho r^{-n-\alpha-1}\int_{B_r}|x_{n+1}|^\alpha|D\varphi_E|\,dr\\
&\qquad+\int_\sigma^\rho r^{-n-\alpha-1}\Psi(E,B_r)\,dr\\
&\qquad+\frac{1}{(n+\alpha)}\liminf_{\varepsilon\to 0}\Bigg[\rho^{-n-\alpha}\int_{B_\rho}|x_{n+1}|^\alpha|D\phi_E^\varepsilon(x)|\,dx\\
&\qquad-\sigma^{-n-\alpha}\int_{B_\sigma}|x_{n+1}|^\alpha|D\phi_E^\varepsilon(x)|\,dx\\
&\qquad+(n+\alpha)\int_\sigma^\rho r^{-n-\alpha-1}\int_{B_r}|x_{n+1}|^\alpha|D\phi_E^\varepsilon(x)|\,dx\,dr\Bigg]\Bigg\}
\end{aligned}$$

which in turn – using the approximation (13) – proves the final estimate

$$\begin{aligned}
&\left(\int_{B_\rho-B_\sigma}|x_{n+1}|^\alpha\,\frac{|D\varphi_E\cdot x|}{|x|^{n+\alpha+1}}\right)^2\le 2\left(\int_{B_\rho-B_\sigma}|x_{n+1}|^\alpha\,\frac{|D\varphi_E|}{|x|^{n+\alpha}}\right)\cdot\\
&\qquad\Bigg\{(n+\alpha)\int_\sigma^\rho r^{-n-\alpha-1}\Psi(E,B_r)\,dr+\rho^{-n-\alpha}\int_{B_\rho}|x_{n+1}|^\alpha|D\varphi_E|\\
&\qquad\qquad-\sigma^{-n-\alpha}\int_{B_\sigma}|x_{n+1}|^\alpha|D\varphi_E|\Bigg\}
\end{aligned}$$

□

Proposition 5 immediately implies the monotonicity formula for minimizing boundaries.

Proposition 6. *Let $\alpha > 0$ and suppose $E \subset \mathbb{R}^{n+1}$ is a Caccioppoli set which locally minimizes $\mathcal{E}$ in $\Omega \subset \mathbb{R}^{n+1}$, i.e. $\Psi(E,\Omega) = 0$. Then we have the inequality*

$$\sigma^{-n-\alpha} \int_{B_\sigma} |x_{n+1}|^\alpha |D\varphi_E| \le \rho^{-n-\alpha} \int_{B_\rho} |x_{n+1}|^\alpha |D\varphi_E|$$

for all balls $B_\sigma = B_\sigma(\xi) \subset B_\rho = B_\rho(\xi) \subset\subset \Omega$, where $\xi = (\xi_1, \dots, \xi_n, 0) \in \mathbb{R}^n \times \{0\}$ is arbitrary.

3. Area growth

Here we suppose that $E \subset \mathbb{R}^{n+1}$ has locally finite perimeter in $\mathbb{R}^{n+1}$ and minimizes

$$\mathcal{E}(U) = \int |x_{n+1}|^\alpha |D\varphi_U|, \quad \alpha > 0$$

locally in $\mathbb{R}^{n+1}$ among Caccioppoli sets, i.e. the *indicator* function

$$\Psi(E,\Omega) = 0$$

for all open sets $\Omega \subset \mathbb{R}^{n+1}$. We say that E has *"sublinear growth"*, if there exists some nonnegative measurable function $s : \mathbb{R}^n \to \mathbb{R}^+$ such that $M = \partial^* E$ fulfills

$$M \subset \{(x, x_{n+1}) \in \mathbb{R}^n \times \mathbb{R}; -s(x) \le x_{n+1} \le s(x)\} \tag{18}$$

and

$$\lim_{R\to\infty} \frac{|s|_{\infty,\mathcal{B}_R(0)}}{R} = 0. \tag{19}$$

Here $\mathcal{B}_R(0) \subset \mathbb{R}^n$ denotes the n-ball with center at $0 \in \mathbb{R}^n$ and $|s|_{\infty,\mathcal{B}_R}$ stands for the sup-norm of s on $\mathcal{B}_R$. Analogously a *function* $u \in BV_{\mathrm{loc}}(\mathbb{R}^n)$ is of *sublinear growth*, if the subgraph

$$U := \{(x,t) \in \mathbb{R}^n \times \mathbb{R}; t < u(x)\}$$

has sublinear growth.

Proposition 7. *Let $E \subset \mathbb{R}^{n+1}$ be a Caccioppoli set which locally minimizes $\mathcal{E}$ in $\mathbb{R}^{n+1}$ for some $\alpha > 0$ and suppose $M = \partial^* E$ is of sublinear growth. Then we have*

$$\lim_{R\to\infty} R^{-n-\alpha} \int_{B_R(0)} |x_{n+1}|^\alpha |D\varphi_E| = 0, \quad B_R(0) \subset \mathbb{R}^{n+1}.$$

Remark. Proposition 7 is sharp as one sees by considering the cones

$$C_n^\alpha := \left\{ (x, x_{n+1}) \in \mathbb{R}^n \times \mathbb{R}; \quad 0 < x_{n+1} < \sqrt{\frac{\alpha}{n-1}} \|x\| \right\}$$

which are of linear growth and minimize

$$\mathcal{E} = \int |x_{n+1}|^\alpha |D\varphi_U|,$$

if – for example – $n = 2$ and $\alpha \geq 6$ say, see [DU1][DU2] for more details. Also, one easily computes

$$\int_{B_R(0)} |x_{n+1}|^\alpha |D\varphi_{C_n^\alpha}| = c(n,\alpha) R^{n+\alpha}$$

for some constant $c(n,\alpha) > 0$.

Proof. Define the cylinder

$$C_R := \{(x, x_{n+1}) \in \mathbb{R}^n \times \mathbb{R}; \quad |x| < R \text{ and } -|s|_{\infty,\mathcal{B}_R} < x_{n+1} < |s|_{\infty,\mathcal{B}_R}\}$$

where $s : \mathbb{R}^n \to \mathbb{R}^+$ is some *"dominance function"* with the properties (18) & (19). The minimum property of E implies for any $\varepsilon > 0$

$$\begin{aligned} \mathcal{E}(E, C_{R+\varepsilon}) :&= \int_{C_{R+\varepsilon}} |x_{n+1}|^\alpha |D\varphi_E| \leq \int_{C_{R+\varepsilon}} |x_{n+1}|^\alpha |D\varphi_{E-\overline{C_R}}| \\ &= \mathcal{E}(E - \overline{C_R}, C_{R+\varepsilon}) \end{aligned} \tag{20}$$

and the trace formula for BV-functions yields for almost all $R, \varepsilon > 0$

$$\mathcal{E}(E - \overline{C_R}, C_{R+\varepsilon}) = \mathcal{E}(E, C_{R+\varepsilon} - \overline{C_R}) + \int_{\partial C_R \cap E} |x_{n+1}|^\alpha d\mathcal{H}_n \tag{21}$$

and similarly also

$$\begin{aligned} \mathcal{E}(E, C_{R+\varepsilon}) &\leq \int_{C_{R+\varepsilon}} |x_{n+1}|^\alpha |D\varphi_{E\cup\overline{C_R}}| \\ &= \mathcal{E}(E \cup \overline{C_R}, C_{R+\varepsilon}) \\ &= \mathcal{E}(E, C_{R+\varepsilon} - \overline{C_R}) + \int_{\partial C_R \cap (\mathbb{R}^{n+1} - E)} |x_{n+1}|^\alpha d\mathcal{H}_n. \end{aligned} \tag{22}$$

(20), (21) and (22) imply the estimate

$$\begin{aligned} \mathcal{E}(E, C_{R+\varepsilon}) = \int_{C_{R+\varepsilon}} &|x_{n+1}|^\alpha |D\varphi_E| \\ \leq\ &\mathcal{E}(E, C_{R+\varepsilon} - \overline{C_R}) \\ &+ \min\left\{\int_{\partial C_R \cap E} |x_{n+1}|^\alpha d\mathcal{H}_n, \int_{\partial C_R \cap (\mathbb{R}^{n+1} - E)} |x_{n+1}|^\alpha d\mathcal{H}_n\right\} \end{aligned}$$

which in turn yields f.a.a. $R > 0$, as $\varepsilon \to 0$

$$\mathcal{E}(E, C_R) \leq \min\left\{\int_{\partial C_R \cap E} |x_{n+1}|^\alpha d\mathcal{H}_n, \int_{\partial C_R \cap (\mathbb{R}^{n+1} - E)} |x_{n+1}|^\alpha d\mathcal{H}_n\right\}. \tag{23}$$

We put $\partial C_R = Z_R \cup D_R^+ \cup D_R^-$, where

$$Z_R := \{(x, x_{n+1}) \in \mathbb{R}^n \times \mathbb{R}; |x| = R \text{ and } -|s|_{\infty,\mathcal{B}_R} \leq x_{n+1} \leq |s|_{\infty,\mathcal{B}_R}\}$$

denotes the vertical wall and

$$D_R^\pm := \{(x, x_{n+1}) \in \mathbb{R}^n \times \mathbb{R}; |x| \leq R, x_{n+1} = \pm|s|_{\infty,\mathcal{B}_R}\}$$

denote the top and bottom of the cylinder ∂C_R respectively. We find the estimate

$$\int_{\partial C_R} |x_{n+1}|^\alpha d\mathcal{H}_n = \int_{D_R^+ \cup D_R^-} |x_{n+1}|^\alpha d\mathcal{H}_n + \int_{Z_R} |x_{n+1}|^\alpha d\mathcal{H}_n$$
$$\leq 2\omega_n R^n |s|^\alpha_{\infty,\mathcal{B}_R} + \frac{\omega_n}{1+\alpha} R^{n-1} |s|^{1+\alpha}_{\infty,\mathcal{B}_R}$$

whence, by virtue of (23) also

$$R^{-n-\alpha} \int_{C_R} |x_{n+1}|^\alpha |D\varphi_E| \leq c(n,\alpha) \left\{ R^{-\alpha} |s|^\alpha_{\infty,\mathcal{B}_R} + R^{-\alpha-1} |s|^{1+\alpha}_{\infty,\mathcal{B}_R} \right\}.$$

Finally, by assumption $M = \partial^* E \subset \{(x, x_{n+1}) \in \mathbb{R}^n \times \mathbb{R}; -s(x) < x_{n+1} < s(x)\}$, whence $M \cap B_R(0) \subset C_R$ and together with (23) and (19) we conclude

$$\lim_{R\to\infty} R^{-n-\alpha} \int_{B_R(0)} |x_{n+1}|^\alpha |D\varphi_E| = 0$$

□

The proof of the following Proposition is standard, see e. g. [GT], chapter 16. For convenience we give the argument in some detail.

Proposition 8. *Let $u \in H^1_{1,\mathrm{loc}}(\mathbb{R}^n - K), K \subset \mathbb{R}^n$ compact, be a weak nonnegative solution of the s.m.s.e. (3) in $(\mathbb{R}^n - K)$ and let $K \subset \mathcal{B}_{R_0}(0) \subset \mathbb{R}^n$. Then for every $\rho > R_0 + 1$ there holds the area estimate*

$$\int_{M\cap B_\rho(0)} x^\alpha_{n+1} d\mathcal{H}_n \leq c(n)\rho^n |u|^\alpha_{\infty,\mathcal{B}_\rho - \mathcal{B}_{R_0+1}} + |u|^\alpha_{\infty,\mathcal{B}_\rho-\mathcal{B}_{R_0+1}} |u|_{1,\mathcal{B}_{R_0+1}-\mathcal{B}_{R_0}}$$

where $M :=$ graph $u_{|\mathcal{B}_\rho - \mathcal{B}_{R_0+1}}$ and $|u|_{p,\Omega}$ denotes the L_p-norm of u on Ω.

Proof. Choose $\rho > R_0 + 1$ and some cut-off function $\eta \in C^{0,1}_c(\mathbb{R}^n - K)$ with the properties

$$\eta(x) = \begin{cases} 1, & \text{if } R_0 + 1 \leq |x| \leq \rho, \\ 0, & \text{if } |x| \leq R_0 \text{ or } |x| \geq 2\rho, \end{cases}$$

and such that a.e.

$$|D\eta| \leq \begin{cases} 1 & \text{for } R_0 \leq |x| \leq R_0 + 1, \\ 0 & \text{for } R_0 + 1 < |x| < \rho, \\ \frac{1}{\rho} & \text{for } \rho \leq |x| \leq 2\rho. \end{cases}$$

Put $\varphi := \eta \cdot u_\rho$, where u_ρ denotes the truncated function

$$u_\rho := \begin{cases} u & \text{on } \{0 \leq u < \rho\}, \\ \rho & \text{on } \{u \geq \rho\}. \end{cases}$$

Then there holds a.e.

$$Du_\rho := \begin{cases} Du & \text{on } \{0 \leq u < \rho\}, \\ 0 & \text{on } \{u \geq \rho\}. \end{cases}$$

and $\varphi \in \mathring{H}_1^1(\mathcal{B}_{2\rho} - K)$ satisfies $D\varphi = D\eta \cdot u_\rho + \eta D u_\rho$ a. e. Upon substitution of φ and $D\varphi$ into the weak formulation of (3)

$$\int_{\mathbb{R}^n - K} \left(\frac{Du\, D\varphi}{\sqrt{1+|Du|^2}} + \frac{\alpha\varphi}{u\sqrt{1+|Du|^2}} \right) dx = 0$$

we arrive at

$$\int_{\mathcal{B}_{2\rho}-\mathcal{B}_{R_0}} \left\{ \frac{Du\, D\eta\, u_\rho}{\sqrt{1+|Du|^2}} + \frac{Du\, Du_\rho \eta}{\sqrt{1+|Du|^2}} + \frac{\alpha\eta u_\rho}{u\sqrt{1+|Du|^2}} \right\} dx = 0.$$

Since $Du_\rho = 0$ on $\{u \geq \rho\}$ a.e. we find

$$\int_{(\mathcal{B}_{2\rho}-\mathcal{B}_{R_0})\cap\{u<\rho\}} \frac{|Du|^2\eta}{\sqrt{1+|Du|^2}}\, dx = -\int_{\mathcal{B}_{2\rho}-\mathcal{B}_{R_0}} \frac{Du\, D\eta\, u_\rho}{\sqrt{1+|Du|^2}}\, dx - \alpha \int_{\mathcal{B}_{2\rho}-\mathcal{B}_{R_0}} \frac{u_\rho \eta}{u\sqrt{1+|Du|^2}}\, dx.$$

In particular, because of $\eta = 1$, if $R_0 + 1 \leq |x| \leq \rho, 0 \leq \eta \leq 1$ and $u, u_\rho \geq 0$ we obtain

$$\int_{(\mathcal{B}_\rho-\mathcal{B}_{R_0+1})\cap\{u<\rho\}} \frac{|Du|^2}{\sqrt{1+|Du|^2}} \leq \int_{\mathcal{B}_{2\rho}-\mathcal{B}_{R_0}} \frac{u_\rho |Du|\, |D\eta|}{\sqrt{1+|Du|^2}}\, dx$$

and hence

$$\int_{(\mathcal{B}_\rho-\mathcal{B}_{R_0+1})\cap\{u<\rho\}} \sqrt{1+|Du|^2}\, dx \leq \mathcal{L}^n(\mathcal{B}_\rho - \mathcal{B}_{R_0+1}) + \int_{\mathcal{B}_{2\rho}-\mathcal{B}_\rho} \frac{u_\rho |Du|\, |D\eta|}{\sqrt{1+|Du|^2}}\, dx + \int_{\mathcal{B}_{R_0+1}-\mathcal{B}_{R_0}} \frac{u_\rho |Du|\, |D\eta|}{\sqrt{1+|Du|^2}}\, dx.$$

Using $0 \leq u_\rho \leq u, 0 \leq u_\rho \leq \rho,\ |D\eta| \leq \frac{1}{\rho}$ on $\{\rho \leq |x| \leq 2\rho\}$ and $|D\eta| \leq 1$ on $\{R_0 \leq |x| \leq R_0 + 1\}$ we find

$$\begin{aligned} &\int_{(\mathcal{B}_\rho-\mathcal{B}_{R_0+1})\cap\{u<\rho\}} \sqrt{1+|Du|^2}\, dx \\ &\leq \mathcal{L}^n(\mathcal{B}_\rho - \mathcal{B}_{R_0+1}) + \mathcal{L}^n(\mathcal{B}_{2\rho} - \mathcal{B}_\rho) + |u|_{1,\mathcal{B}_{R_0+1}-\mathcal{B}_{R_0}} \\ &\leq c_1(n)\rho^n + |u|_{1,\mathcal{B}_{R_0+1}-\mathcal{B}_{R_0}}. \end{aligned}$$

Thus we have

$$\int_{(\mathcal{B}_\rho-\mathcal{B}_{R_0+1})\cap\{u<\rho\}} u^\alpha \sqrt{1+|Du|^2}\, dx \leq |u|^\alpha_{\infty,\mathcal{B}_\rho-\mathcal{B}_{R_0+1}} \{c_1(n)\rho^n + |u|_{1,\mathcal{B}_{R_0+1}-\mathcal{B}_{R_0}}\}$$

and in particular, with $M = \operatorname{graph} u_{|\mathcal{B}_\rho-\mathcal{B}_{R_0+1}}$

$$\int_{M\cap B_\rho(0)} x^\alpha_{n+1} d\mathcal{H}_n \leq c_1(n)\rho^n |u|^\alpha_{\infty,\mathcal{B}_\rho-\mathcal{B}_{R_0+1}} + |u|^\alpha_{\infty,\mathcal{B}_\rho-\mathcal{B}_{R_0+1}} |u|_{1,\mathcal{B}_{R_0+1}-\mathcal{B}_{R_0}}.$$

□

4. Proofs

We start with

Proof of Theorem 1. Suppose on the contrary to the statement of Theorem 1, there is a Lipschitz-solution $u \geq 0$ of the s.m.s.e. $(*)$ which satisfies the growth condition

$$u(x) = \mathrm{o}(|x|) \quad \text{as} \quad |x| \to \infty.$$

By Propositions 3 and 4, especially formula (12) applied to $M = \operatorname{graph}(u), d\mu = d\mathcal{H}_n$ and $\xi = 0 \in \mathbb{R}^{n+1}$ we get for all $0 < \sigma < \rho < \infty$ the inequality

$$\sigma^{-n-\alpha} \int_{B_\sigma(0)\cap M} x_{n+1}^\alpha d\mathcal{H}^n \leq \rho^{-n-\alpha} \int_{B_\rho(0)\cap M} x_{n+1}^\alpha d\mathcal{H}^n.$$

Since $\mathcal{L}^n\left(\{u = 0\}\right) = 0$ there is some $\sigma_0 > 0$ with

$$\sigma_0^{-n-\alpha} \int_{B_{\sigma_0}\cap M} x_{n+1}^\alpha d\mathcal{H}^n > 0.$$

However, according to Proposition 8 we must have

$$\lim_{\rho\to\infty} \rho^{-n-\alpha} \int_{B_\rho\cap M} x_{n+1}^\alpha d\mathcal{H}^n = 0,$$

an obvious contradiction. □

Proof of Theorem 2. Let $u \in BV_{+,\mathrm{loc}}^{1+\alpha}(\mathbb{R}^n)$ be a local minimum of the variational integral

$$E = \int u^\alpha \sqrt{1 + |Du|^2}\,, \alpha > 0$$

in the class $BV_+^{1+\alpha}(\Omega)$, $\Omega \subset \mathbb{R}^n$ arbitrary. Then we have $u \in BV_{\mathrm{loc}}(\mathbb{R}^n)$ (in fact $u \in H_{1,\mathrm{loc}}^1(\mathbb{R}^n)$ according to Tennstädt [TT1]) and the subgraph

$$U := \left\{(x,t) \in \mathbb{R}^{n+1}; t < u(x)\right\}$$

has locally finite perimeter in $\mathbb{R}^{n+1}$. By Theorem 10 in [BD], the supgraph U locally minimizes

$$\mathcal{E}(U) = \int |x_{n+1}|^\alpha |D\varphi_U|$$

in $\mathbb{R}^{n+1}$. (In fact, in the paper [BD] only the case $\alpha = 1$ is considered, however the generalization to arbitrary $\alpha > 0$ is straight forward!). Now we are in the situation described in Proposition 6 with minimizing set U and arbitrary open set $\Omega \subset \mathbb{R}^{n+1}$. For $\xi = 0$ and $0 < \sigma < \rho < \infty$ arbitrary we get

$$\sigma^{-n-\alpha} \int_{B_\rho} |x_{n+1}|^\alpha |D\varphi_U| \leq \rho^{-n-\alpha} \int_{B_\rho(0)} |x_{n+1}|^\alpha |D\varphi_U|.$$

By virtue of Proposition 7 and by letting $\rho \to \infty$ we finally arrive at

$$\int_{B_\sigma(0)} |x_{n+1}|^\alpha |D\varphi_U| = 0$$

for every $\sigma > 0$, hence $\partial U = \{x_{n+1} = 0\}$. □

Proof of Theorem 3. Theorem 3 follows from Propositions 6 and 7 analogously to the proof to Theorem 2. □

Proof of Theorem 4. Suppose on the contrary to the statement of Theorem 4, that there is a non-trivial $u \in H_{1,\mathrm{loc}}^1(\mathbb{R}^n) \cap C^0(\mathbb{R}^n)$ which solves the s.m.s.e. weakly in

$\mathbb{R}^n - \{u = 0\}$ and which is of sublinear growth. By Proposition 3′ $M = \text{graph}(u)$ is stationary in $\mathbb{R}^{n+1}$. Proposition 4, formula (12) with $\xi = 0$, Proposition 8, and the assumption of sublinear growth imply that

$$\sigma^{-n-\alpha} \int_{B_\sigma(0)\cap M} x_{n+1}^\alpha d\mathcal{H}_n = 0$$

for every $\sigma > 0$ and $M = \text{graph}(u) \subset \mathbb{R}^{n+1}$; whence we had $u = 0$ on $\mathbb{R}^n$. This contradiction finishes the proof of Theorem 4. □

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